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The Landau–Lifshitz equation and related models

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Abstract

In this manuscript, I present my research on nonlinear dispersive PDEs, done after my Ph.D. Most of the results concern the Landau–Lifshitz equation, which is a quasilinear equation that describes the evolution of the magnetization vector in ferromagnetic materials. This equation is related, depending on the anisotropy, dispersion and dissipation, to the Shrödinger map equation, the heat flow for harmonic maps, and the Gross–Pitaevskii equation.

There are several ways to gain a better understanding of the dynamics of a PDE. With my collaborators, we focused on the study of particular solutions and on the relation with other equations in some asymptotics regimes for the Landau–Lifshitz equation. Regarding the particular solutions, we studied properties of solitons (traveling waves) and self-similar solutions (forward and backward), such as their existence and stability. Concerning the asymptotics regimes for the anisotropic Landau–Lifshitz equation, we established the connection with the Sine–Gordon equation in the case of a strong easy-plane anisotropy, and with the cubic Schrödinger equation in the presence of a strong easy-axis anisotropy. In addition, we tackled some issues related to the Cauchy problem to provide a clear framework for our results.

In the last chapter, we also studied the Gross–Pitaevskii equation including nonlocal effects in the potential energy. In particular, we provided some results concerning the existence and stability of solitons for this equation.

Résumé

Dans ce manuscrit je présente mes recherches sur des EDP dispersives non linéaires, faites après mon doctorat. La plupart des résultats concernent l'équation de Landau–Lifshitz, qui est une équation quasi-linéaire qui décrit l'évolution du vecteur de magnétisation dans les matériaux ferromagnétiques. Cette équation est liée, en fonction de l'anisotropie, de la dispersion et de la dissipation, à plusieurs équations telles que l'équation de Shrödinger maps, l'équation de la chaleur pour les applications harmoniques et l'équation de Gross–Pitaevskii.

Il existe plusieurs façons de mieux comprendre la dynamique d'une EDP. Avec mes collaborateurs, nous nous sommes concentrés sur l'étude de solutions particulières et sur la relation avec d'autres équations dans certains régimes asymptotiques pour l'équation de Landau–Lifshitz. En ce qui concerne les solutions particulières, nous avons étudié les propriétés des solitons (ondes progressives) et des solutions auto-similaires (*forward* et *backward*), telles que leur existence et stabilité. Concernant les régimes asymptotiques pour l'équation Landau–Lifshitz anisotropique, nous avons établi le lien avec l'équation de Sine–Gordon dans le cas d'une forte anisotropie planaire, et avec l'équation de Schrödinger cubique en présence d'une forte anisotropie axiale. De plus, nous avons abordé certaines questions liées au problème de Cauchy afin de préciser le cadre de nos résultats.

Dans le dernier chapitre, nous avons également étudié l'équation de Gross–Pitaevskii, prenant en compte des effets non locaux dans l'énergie potentielle. En particulier, nous avons fourni quelques résultats concernant l'existence et la stabilité des solitons pour cette équation.

Contents

Introduction

In this work, I present several mathematical results concerning nonlinear PDEs of Schrödinger type, done after my Ph.D., defended in 2011 under the supervision of Fabrice Béthuel. My research mainly focus on the dynamics of these equations and on the existence and behavior of particular solutions, such as solitons and self-similar solutions.

Most of the results in this thesis concern to the Landau–Lifshitz equation (LL), describing the magnetization vector in ferromagnetic materials. The LL equation is a nonlinear PDE taking values in the sphere $\mathbb{S}^2$, and is related to several classical equations. There are many variations of the LL equation, and I refer to them simply by LL, and sometimes by LLG to emphasis the effect of a damping Gilbert term. For instance, in the undamped case, it is a dispersive equation and shares several properties with nonlinear Schrödinger equations with nontrivial boundary conditions at infinity, such as the Gross–Pitaevskii equation. In the presence of damping, the LL equation can be seen as parabolic quasilinear system, related to the complex Ginzburg–Landau equation. The LL equation is also related to the Sine–Gordon equation, the harmonic map flow and the Localized Induction Approximation. Since the LL equation is less well-known that other PDEs, I give preamble to introduce precisely the LL equation, as well as some terminology, transformations, and some examples of explicit solutions.

In the Chapter 1, I provide some results concerning the local well-posedness for the LL equation in Sobolev spaces, in the pure dispersive case, i.e. without damping. Most of the results in the literature in this framework consider the isotropic case, i.e. the Schrödinger map equation, but it is not always possible to adapt these results to include anisotropic perturbations. In this chapter, I review known results and provide an alternative proof for local well-posedness for smooth solutions introducing high order energy quantities with good symmetrization properties. This chapter is based on the papers [dLG15a, dLG18].

Chapter 2 is devoted explain the results obtained in collaboration with P. Gravejat in [dLG15a, dLG18] concerning some asymptotic regimes for the anisotropic LL equation. Indeed, using formal arguments, Sklyanin [Skl79] derived (in dimension one) two asymptotic regimes corresponding to the Sine–Gordon equation and the cubic Schrödinger equation. We provide a mathematical framework to rigorously prove the connection between these equations and we also give estimates of the error in Sobolev norms.

In our collaboration with P. Gravejat in [dLG15a, dLG16], we have also investigated the stability of solitons for the LL equation with easy-plane anisotropy in dimension one. This result is presented in Chapter 3, we describe the main result that establishes that sums of solitons are orbitally stable, provided that the (nonzero) speeds of the solitons are different, and that their initial positions are sufficiently separated and ordered according to their speeds. We will also discuss their asymptotic stability, obtained later during the Ph.D. of Y. Bahri.

Chapter 4 is dedicated to explain the results obtained in collaboration with S. Gutiérrez in [dLG15b, dLG19, dLG20], regarding the isotropic LLG equation. The aim is to study the existence and properties of forward and backward self-similar solutions, i.e. of expanders and shrinkers. The expanders provide a family of global solutions to the LLG equation with discontinuous initial data that are smooth and have finite energy for all positive times, while the shrinkers are explicit examples of smooth solutions blowing up in finite time. Furthermore, we prove a global well-posedness result for the LLG equation, provided that the BMO semi-norm of the initial data is small. As a consequence, we deduce that the aforementioned expanders are stable (in some sense), and the existence of self-similar forward solutions in any dimension.

Finally, in Chapter 5, I discuss some results concerning nonlocal effects in the potential energy for the Gross–Pitaevskii equation, for a variety of nonlocal interactions. I briefly summarize necessary conditions on the potential modeling the nonlocal interaction, in order to generalize known properties for the contact interaction given by a Dirac delta function, that I obtained during my PhD [dL10, dL09]. In particular, I tackle the global well-posedness and the nonexistence of traveling waves for supersonic speeds. Afterwards, I describe a result in collaboration with P. Mennuni in [dLM20], providing necessary conditions on the interaction to have the existence of a branch of orbital stable traveling waves solutions, with nonvanishing conditions at infinity.

List of articles

Articles done after my Ph.D.

[dLG15a] A. de Laire and P. Gravejat. Stability in the energy space for chains of solitons of the Landau-Lifshitz equation. *J. Differential Equations*, 258(1):1-80, 2015.

[dLG15b] A. de Laire and S. Gutiérrez. Self-similar solutions of the one-dimensional Landau-Lifshitz-Gilbert equation. *Nonlinearity*, 28(5):1307-1350, 2015.

[dLG18] A. de Laire and P. Gravejat. The Sine-Gordon regime of the Landau-Lifshitz equation with a strong easy-plane anisotropy. *Ann. Inst. Henri Poincaré, Analyse Non Linéaire*, 35(7):1885-1945, 2018.

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[dLG20] A. de Laire and S. Gutiérrez. Self-similar shrinkers of the one-dimensional Landau-Lifshitz-Gilbert. Preprint.

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[dL09] A. de Laire. Non-existence for travelling waves with small energy for the Gross-Pitaevskii equation in dimension $N \geq 3$. *C. R. Math. Acad. Sci. Paris*, 347(7-8):375-380, 2009.

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[dL16] A. de Laire and P. Gravejat. Stabilité des solitons de l'équation de Landau- Lifshitz à anisotropie planaire. In *Séminaire Laurent Schwartz – Équations aux dérivées partielles et applications. Année 2014–2015*, pages Exp. No. XVII, 27. Ed. Éc. Polytech., Palaiseau, 2016.

Preamble

The Landau–Lifshitz equation

The Landau–Lifshitz (LL) equation has been introduced in 1935 by L. Landau and E. Lifshitz in [LL35] and it constitutes nowadays a fundamental tool in the magnetic recording industry, due to its applications to ferromagnets [Wei12]. This PDE describes the dynamics of the orientation of the magnetization (or spin) in ferromagnetic materials, and it is given by

$$\partial_t \boldsymbol{m} + \boldsymbol{m} \times H_{\text{eff}}(\boldsymbol{m}) = 0, \tag{1}$$

where $\boldsymbol{m} = (m_1, m_2, m_3) : \mathbb{R}^N \times I \longrightarrow \mathbb{S}^2$ is the spin vector, $I \subset \mathbb{R}$ is a time interval, $\times$ denotes the usual cross-product in $\mathbb{R}^3$, and $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$. Here $H_{\text{eff}}(\boldsymbol{m})$ is the effective magnetic field, corresponding to (minus) the L^2-derivative of the magnetic energy of the material. We will focus on energies of the form

$$E_{\text{LL}}(\boldsymbol{m}) = E_{\text{ex}}(\boldsymbol{m}) + E_{\text{ani}}(\boldsymbol{m}),$$

where the *exchange* energy

$$E_{\text{ex}}(\boldsymbol{m}) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \boldsymbol{m}|^2 = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla m_1|^2 + |\nabla m_2|^2 + |\nabla m_3|^2,$$

accounts for the local tendency of $\boldsymbol{m}$ to align the magnetization field, and the *anisotropy* energy

$$E_{\text{ani}}(\boldsymbol{m}) = \frac{1}{2}\int_{\mathbb{R}^N} \langle \boldsymbol{m}, J\boldsymbol{m}\rangle_{\mathbb{R}^3}, \qquad J \in \operatorname{Sym}_3(\mathbb{R}),$$

accounts for the likelihood of $\boldsymbol{m}$ to attain one or more directions of magnetization, which determines the *easy* directions. Due to the invariance of (1) under rotations[1], we can assume that J is a diagonal matrix $J := \operatorname{diag}(J_1, J_2, J_3)$, and thus the anisotropy energy reads

$$E_{\text{ani}}(\boldsymbol{m}) = \frac{1}{2}\int_{\mathbb{R}^N} (\lambda_1 m_1^2 + \lambda_3 m_3^2), \tag{2}$$

[1] In fact, using that

$$(M\boldsymbol{a}) \times (M\boldsymbol{b}) = (\det M)(M^{-1})^T(\boldsymbol{a} \times \boldsymbol{b}), \qquad \text{for all } M \in \mathcal{M}_{3,3}(\mathbb{R}),\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3,$$

it is easy to verify that if $\boldsymbol{m}$ is a solution of (1), then so is $R\boldsymbol{m}$, for any $R \in SO(3)$.

with $\lambda_1 := J_2 - J_1$ and $\lambda_3 := J_2 - J_3$. Therefore (1) can be recast as

$$\partial_t \boldsymbol{m} + \boldsymbol{m} \times (\Delta \boldsymbol{m} - \lambda_1 m_1 \boldsymbol{e}_1 - \lambda_3 m_3 \boldsymbol{e}_3) = 0, \tag{3}$$

where $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is the canonical basis of $\mathbb{R}^3$. Notice that for finite energy solutions, (2) formally implies that $m_1(x) \to 0$ and $m_3(x) \to 0$, as $|x| \to \infty$, and hence

$$|m_2(x)| \to 1, \quad \text{as } |x| \to \infty.$$

For biaxial ferromagnets, all the numbers J_1, J_2 and J_3 are different, so that $\lambda_1 \neq \lambda_3$ and $\lambda_1\lambda_3 \neq 0$. Uniaxial ferromagnets are characterized by the property that only two of the numbers J_1, J_2 and J_3 are equal. For instance, the case $J_1 = J_2$ corresponds to $\lambda_1 = 0$ and $\lambda_3 \neq 0$, so that the material has a uniaxial anisotropy in the direction $\boldsymbol{e}_3$. Hence, the ferromagnet owns an *easy-axis* anisotropy along the vector $\boldsymbol{e}_3$ if $\lambda_3 < 0$, while the anisotropy is *easy-plane* along the plane $x_3 = 0$ if $\lambda_3 > 0$. Finally, in the isotropic case $\lambda_1 = \lambda_3 = 0$, equation (3) reduces to the well-known Schrödinger map equation

$$\partial_t \boldsymbol{m} + \boldsymbol{m} \times \Delta \boldsymbol{m} = 0. \tag{4}$$

The LL equation (3) is a nonlinear dispersive PDE. Indeed, let us consider a small perturbation of the constant solution $\boldsymbol{e}_2$ of the form

$$\boldsymbol{m} = \frac{\boldsymbol{e}_2 + s\boldsymbol{v}}{|\boldsymbol{e}_2 + s\boldsymbol{v}|},$$

for s small, where $\boldsymbol{v} = (v_1, v_2, v_3)$. Using that $\boldsymbol{m} = \boldsymbol{e}_2 + s(v_1\boldsymbol{e}_1 + v_3\boldsymbol{e}_3) + \mathcal{O}(s^2)$ and dropping the terms in s^2, we obtain the linearized system for $\boldsymbol{v}$

$$\begin{aligned}
\partial_t v_1 + \Delta v_3 - \lambda_3 v_3 &= 0,\\
\partial_t v_3 - \Delta v_1 + \lambda_1 v_1 &= 0,
\end{aligned}$$

so that

$$\partial_{tt} v_1 + \Delta^2 v_1 - (\lambda_1 + \lambda_3)\Delta v_1 + \lambda_1\lambda_3 v_1 = 0.$$

Thus, we get the dispersion relation

$$\omega(k) = \pm\sqrt{|k|^4 + (\lambda_1 + \lambda_3)|k|^2 + \lambda_1\lambda_3}, \tag{5}$$

for linear sinusoidal waves of frequency ω and wavenumber k, i.e. solutions of the form $e^{i(k\cdot x - \omega t)}$. In particular, the group velocity is given by

$$\nabla\omega(k) = \pm\frac{2|k|^2 + \lambda_1 + \lambda_3}{\sqrt{(|k|^2 + \lambda_1)(|k|^2 + \lambda_3)}}k.$$

From (5), we can recognize similarities with some classical dispersive equations. For instance, for the Schrödinger equation

$$i\partial_t \psi + \Delta\psi = 0,$$

the dispersion relation is $\omega(k) = |k|^2$, corresponding to $\lambda_1 = \lambda_3 = 0$ in (5), i.e. the Schrödinger maps equation (4).

When considering Schrödinger equations with nonvanishing conditions at infinity, the typical example is the Gross–Pitaesvkii equation

$$i\partial_t\psi + \Delta\psi + \sigma\psi(1 - |\psi|^2) = 0, \tag{6}$$

and the dispersion relation for the linearized equation at the constant solution equal to 1 is

$$\omega(k) = \pm\sqrt{|k|^4 + 2\sigma|k|^2},$$

for $\sigma > 0$, that corresponds to take $\lambda_1 = 0$ or $\lambda_3 = 0$, with $\lambda_1 + \lambda_3 = 2\sigma$, in (5).

Finally, let us consider the Sine–Gordon equation

$$\partial_{tt}\psi - \Delta\psi + \sigma\sin(\psi) = 0,$$

$\sigma > 0$, whose linearized equation at 0 is given by the Klein–Gordon equation, with dispersion relation

$$\omega(k) = \pm\sqrt{|k|^2 + \sigma},$$

that behaves like (5) for $\lambda_1\lambda_3 = \sigma$ and $\lambda_1 + \lambda_3 = 1$, at least for k small.

In this context, the Landau–Lifshitz equation is considered as a universal model from which it is possible to derive other completely integrable equations. We will provide some rigorous results in this context in Chapter 2.

§1 Solitons

In dimension one, the LL equation is completely integrable by means of the inverse scattering method (see e.g. [FT07]) and, using this technique, explicit solitons and multisolitons solutions can be constructed (see e.g. [BBI14]).

We will define a soliton for the LL equation (3) as a traveling wave propagating with speed c along the x_1-axis, i.e. of the form

$$\boldsymbol{m}(x,t) = \boldsymbol{m}_c(x_1 - ct, x_2, \dots, x_N).$$

By substituting this formula in (3), taking cross product with $\boldsymbol{m}_c$ and noticing that for a (smooth) function satisfying $|\boldsymbol{v}| = 1$, we have[2]

$$\boldsymbol{v} \times (\boldsymbol{v} \times \Delta\boldsymbol{v}) = \Delta\boldsymbol{v} + |\nabla\boldsymbol{v}|^2\boldsymbol{v}, \tag{7}$$

we obtain the equation for the profile $\boldsymbol{m}_c = (m_{1,c}, m_{2,c}, m_{3,c})$,

$$\Delta\boldsymbol{m}_c + |\nabla\boldsymbol{m}_c|^2\boldsymbol{m}_c + (\lambda_1 m_{1,c}^2 + \lambda_3 m_{3,c}^2)\boldsymbol{m}_c - (\lambda_1 m_{1,c}\boldsymbol{e}_1 + \lambda_3 m_{3,c}\boldsymbol{e}_3) + c\,\boldsymbol{m}_c \times \partial_1\boldsymbol{m}_c = 0. \tag{8}$$

[2]Here we use also the identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$, for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$.

Assuming without loss of generality, that $\lambda_3 > \lambda_1 > 0$, we set $c^* := \lambda_3^{1/2} - \lambda_1^{1/2}$. Then, in the one-dimensional case $N = 1$, for $|c| \leq c^*$, nonconstant solitons $\boldsymbol{m}_c$ satisfying the boundary conditions at infinity

$$\boldsymbol{m}_c(-\infty) = (0, -1, 0) \quad \text{and} \quad \boldsymbol{m}_c(\infty) = (0, 1, 0), \tag{9}$$

are explicitly given by the formulas

$$\boldsymbol{m}_c^{\pm}(x) = \Big(\frac{a_c^{\pm}}{\cosh(\mu_c^{\pm} x)}, \tanh(\mu_c^{\pm} x), \frac{(1 - (a_c^{\pm})^2)^{\frac{1}{2}}}{\cosh(\mu_c^{\pm} x)} \Big), \tag{10}$$

up to the geometric invariances of the equation, which are the translations and the orthogonal symmetries with respect to the axes $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. In this formula, the values of $a_c^{\pm}$ and $\mu_c^{\pm}$ are given by

$$a_c^{\pm} := \delta_c \Big(\frac{c^2 + \lambda_3 - \lambda_1 \mp \big((\lambda_3 + \lambda_1 - c^2)^2 - 4\lambda_1\lambda_3 \big)^{\frac{1}{2}}}{2(\lambda_3 - \lambda_1)} \Big)^{\frac{1}{2}},$$

and

$$\mu_c^{\pm} = \Big(\frac{\lambda_3 + \lambda_1 - c^2 \pm \big((\lambda_3 + \lambda_1 - c^2)^2 - 4\lambda_1\lambda_3 \big)^{\frac{1}{2}}}{2} \Big)^{\frac{1}{2}},$$

with $\delta_c = 1$, if $c \geq 0$, and $\delta_c = -1$, when $c < 0$. Noticing that

$$(\mu_c^{\pm})^2 = \lambda_1 (a_c^{\pm})^2 + \lambda_3 (1 - (a_c^{\pm})^2),$$

we get that the energy of the solitons $m_c^{\pm}$ is equal to

$$E_{\rm LL}(\boldsymbol{m}_c^{\pm}) = 2\mu_c^{\pm}.$$

In this manner, the solitons form two branches in the plane $(c, E_{\rm LL})$. The lower branch

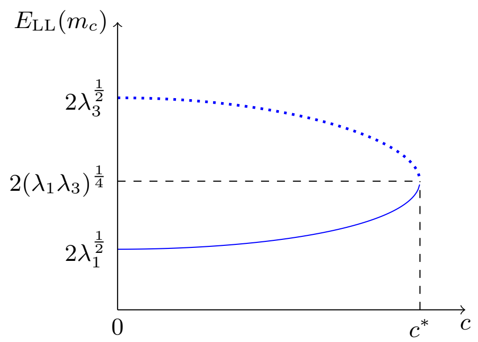


Figure 1: The curves $E_{\rm LL}(\boldsymbol{m}_c^+)$ and $E_{\rm LL}(\boldsymbol{m}_c^-)$ in dotted and solid lines, respectively.

corresponds to the solitons $\boldsymbol{m}_c^-$, and the upper one to the solitons $\boldsymbol{m}_c^+$ as depicted in Figure 1. The lower branch is strictly increasing and convex with respect to $c \in [0, c^*]$, with

$$E(\boldsymbol{m}_0^-) = 2\lambda_1^{\frac{1}{2}} \quad \text{and} \quad E(\boldsymbol{m}_{c^*}^-) = 2(\lambda_1\lambda_3)^{\frac{1}{4}}.$$

The upper branch is a strictly decreasing and concave function of $c \in [0, c^*]$, with

$$E(\boldsymbol{m}_0^+) = 2\lambda_3^{\frac{1}{2}} \quad \text{and} \quad E(\boldsymbol{m}_{c^*}^+) = 2(\lambda_1\lambda_3)^{\frac{1}{4}},$$

and the two branches meet at the common soliton $\boldsymbol{m}_{c^*}^- = \boldsymbol{m}_{c^*}^+$.

In the limit $\lambda_1 \to 0$, the lower branch vanishes, while the upper one goes to the branch of solitons for the Landau-Lifshitz equation with easy-plane anisotropy (see e.g. [dL12]).

Finally, we remark that since a_c^- is a strictly decreasing and a continuous function of c, with

$$a_0^- = 1 \quad \text{and} \quad a_{c^*}^- = \Big(\frac{\sqrt{\lambda_3}}{\sqrt{\lambda_1} + \sqrt{\lambda_3}}\Big)^{\frac{1}{2}},$$

the function $\check{m}_c^- := m_{1,c}^- + i m_{2,c}^-$ may always be lifted as

$$\check{m}_c^- = \sqrt{1 - (m_{3,c}^-)^2}\big(\sin(\varphi_c^-) + i\cos(\varphi_c^-)\big),$$

with

$$\varphi_c^-(x) = 2\arctan\left(\frac{\big((a_c^-)^2 + \sinh^2(\mu_c^- x)\big)^{\frac{1}{2}} - \sinh(\mu_c^- x)}{a_c^-}\right).$$

In particular, when $c = 0$, we get

$$\varphi_0^-(x) = 2\arctan\left(e^{-\sqrt{\lambda_1}x}\right),$$

that corresponds to the (stationary) anti-kink solution to the Sine–Gordon equation

$$\partial_{tt}\psi - \partial_{xx}\psi + \frac{\lambda_1}{2}\sin(2\psi) = 0.$$

In addition to the explicit solitons satisfying the boundary condition (9), it is also possible to obtain solitons with the same limit at $\pm\infty$, e.g. $\boldsymbol{m}_c(\pm\infty) = (0, 1, 0)$ (see [dLG21]). As mentionated before, multisolitons can also be constructed by the inverse scatttttering method [BBI14].

In the higher dimensional case $N \in \{2, 3\}$, N. Papanicolaou and P. N. Spathis [PS99] found finite energy solitons to (8) in the easy-plane case $\lambda_1 = 0$, by using formal developments and numerical simulations. More precisely, they obtained a branch of solitons, for all $|c| \leq \sqrt{\lambda_3}$. Later, F. Lin and J. Wei [LW10] proved the existence of these solitons for small values of c by perturbative arguments.

§2 The hydrodynamical formulations

In the seminal work [Mad26], Madelung shows that the nonlinear Schrödinger equation (NLS) can be recast into the form of a hydrodynamic system. For instance, for the NLS equation

$$i\partial_t\Psi + \Delta\Psi = \Psi f(|\Psi|^2) = 0,$$

assuming that $\rho := |\Psi|^2$ does not vanish, the Madelung transform

$$\psi = \sqrt{\rho}e^{i\phi}$$

leads to the system

$$\begin{aligned} &\partial_t\rho + 2\,\mathrm{div}(\rho\nabla\phi) = 0, \\ &\partial_t\phi + |\nabla\phi|^2 + f(\rho) = \frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}}. \end{aligned} \tag{11}$$

Therefore, setting $\boldsymbol{v} = 2\nabla\phi$, we get the Euler–Korteweg system

$$\begin{aligned} &\partial_t\rho + \mathrm{div}(\rho\boldsymbol{v}) = 0, \\ &\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + 2\nabla(f(\rho)) = 2\nabla\Big(\frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}}\Big), \end{aligned}$$

which is a dispersive perturbation of the classical equation Euler equation for compressible fluids, with the additional term $2\nabla(\Delta(\sqrt{\rho})/\sqrt{\rho}$, which is interpreted as quantum pressure in the quantum fluids models [CDS12, BGLV19].

The Madelung transform is useful to study properties of NLS equations with nonvanishing conditions at infinity (see [BGS14, CR10]). In the case of the Gross–Pitaesvkii equation (6), we have $f(\rho) = \rho - 1$, so setting $u = 1 - |\psi|^2 = 1 - \rho$, we can write the hydrodynamical system (11) as

$$\begin{aligned} &\partial_t u = 2\mathrm{div}\big((1-u)\nabla\phi\big), \\ &\partial_t\phi = \sigma u - |\nabla\phi|^2 - \frac{\Delta u}{2(1-u)} - \frac{|\nabla u|^2}{4(1-u)^2}. \end{aligned} \tag{12}$$

Coming back to the LL equation (3), let $\boldsymbol{m}$ be a solution of this equation such that the map $\check{m} := m_1 + im_2$ does not vanish. In the spirit of the Madelung transform, we set

$$\check{m} = \sqrt{1 - m_3^2}e^{-i\phi},$$

so that we obtain the following hydrodynamical system in terms of the variables $u := m_3$ and ϕ,

$$\begin{aligned} &\partial_t u = \mathrm{div}\,\big((1-u^2)\nabla\phi\big) + \frac{\lambda_1}{2}(1-u^2)\sin(2\phi), \\ &\partial_t\phi = \lambda_3 u - u|\nabla\phi|^2 - \frac{\Delta u}{1-u^2} - \frac{u|\nabla u|^2}{(1-u^2)^2} - \lambda_1 u\cos^2(\phi). \end{aligned} \tag{13}$$

Therefore, the hydrodynamical formulations (12) and (13) are very similar when $\lambda_1 = 0$. As shown in the next chapters, these kinds of formulations will be essential in the study some solutions of the LL equation.

§3 The dissipative model

In 1955, T. Gilbert proposed in [Gil55] a modification of equation (1) to incorporate a damping term. The so-called Landau–Lifshit–Gilbert (LLG) equation then reads

$$\partial_t\boldsymbol{m} = \beta\boldsymbol{m}\times H_{\mathrm{eff}}(\boldsymbol{m}) - \alpha\boldsymbol{m}\times(\boldsymbol{m}\times H_{\mathrm{eff}}(\boldsymbol{m})), \tag{14}$$

where $\beta \geq 0$ and $\alpha \geq 0$, so that there is dissipation when $\alpha > 0$, and in that case we refer to α as the Gilbert damping coefficient. Note that, by performing a time scaling, we assume w.l.o.g. that

$$\alpha \in [0,1] \quad \text{and} \quad \beta = \sqrt{1-\alpha^2}.$$

Thus, using (7), we see that in the limit case $\beta = 0$ (and so $\alpha = 1$), the LLG equation reduces to the heat-flow equation for harmonic maps

$$\partial_t \boldsymbol{m} - \Delta \boldsymbol{m} = |\nabla \boldsymbol{m}|^2 \boldsymbol{m}. \tag{HFHM}$$

This classical equation is an important model in several areas such as differential geometry and calculus of variations. It is also related with other problems such as the theory of liquid crystals and the Ginzburg–Landau equation. For more details, we refer to the surveys [EL78, EL88, Hél02, LW08, Str96].

As in previous sections, one way to start the study of the LLG is noticing the link with other PDEs. Let us illustrate this point in the isotropic case $H_{\text{eff}}(\boldsymbol{m}) = \Delta \boldsymbol{m}$, i.e. for

$$\partial_t \boldsymbol{m} = \beta \boldsymbol{m} \times \Delta \boldsymbol{m} - \alpha \boldsymbol{m} \times (\boldsymbol{m} \times \Delta \boldsymbol{m}). \tag{15}$$

For a smooth solution $\boldsymbol{m}$ with $m_3 > -1$, we can use the stereographic projection

$$u = \mathcal{P}(\boldsymbol{m}) := \frac{m_1 + i m_2}{1 + m_3},$$

that satisfies the quasilinear Schrödinger equation

$$i u_t + (\beta - i\alpha)\Delta u = 2(\beta - i\alpha)\frac{\bar{u}(\nabla u)^2}{1 + |u|^2},$$

where we used the notation $(\nabla u)^2 = \nabla u \cdot \nabla u = \sum_{j=1}^N (\partial_{x_j} u)^2$ (see e.g. [LN84] for details). When $\alpha > 0$, one can use the properties of the semigroup $e^{(\alpha + i\beta)t\Delta}$ to establish a Cauchy theory for rough initial data (see e.g. [dLG19, Mel12] and the references therein).

When $N = 1$, the LL equation is also related to the Localized Induction Approximation (LIA), also called *binormal flow*, a geometric curve flow modeling the self-induced motion of a vortex filament within an inviscid fluid in $\mathbb{R}^3$ [Lak11]. This is related with the geometric representation of the LL equation as follows. Let us suppose that $\boldsymbol{m}$ is the tangent vector of a curve in $\mathbb{R}^3$, that is $\boldsymbol{m}(x,t) = \partial_x \boldsymbol{X}(x,t)$, for some curve $\boldsymbol{X}(x,t) \in \mathbb{R}^3$ parameterized by arclenght[3]. When the curvature $\mathrm{k}(x,t)$ and the torsion $\tau(x,t)$ of the curve $\boldsymbol{X}$ are specified (and assuming that $\boldsymbol{X}$ has nonvanishing curvature), the Serret–Frenet system

$$\begin{aligned} \partial_x \boldsymbol{m} &= \mathrm{k}\boldsymbol{n}, \\ \partial_x \boldsymbol{n} &= -\mathrm{k}\boldsymbol{m} + \tau \boldsymbol{b}, \\ \partial_x \boldsymbol{b} &= -\tau \boldsymbol{n}, \end{aligned} \tag{16}$$

[3]Notice that since $\boldsymbol{m} \in \mathbb{S}^2$, this condition is compatible with the arclength parametrization $|\partial_x \boldsymbol{X}| = 1$.

determines uniquely the shape of the curve, and in particular the space evolution of the tangent vector $\boldsymbol{m} = \partial_x \boldsymbol{X}$. Using (16), we get

$$\partial_{xx}\boldsymbol{m} = \partial_x \mathrm{k}\boldsymbol{n} + \mathrm{k}(-\mathrm{k}\boldsymbol{n} + \tau\boldsymbol{b}),$$

and thus the LLG equation (15) rewrites as

$$\partial_t \boldsymbol{m} = \beta(\partial_x \mathrm{k}\boldsymbol{b} - \mathrm{k}\tau\boldsymbol{n}) + \alpha(\mathrm{k}\tau\boldsymbol{b} + \partial_x\mathrm{k}\boldsymbol{n}) \tag{17}$$

in terms of intrinsic quantities k, τ and the Serret-Frenet trihedron $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{b})$. From the geometric representation of the LLG equation (17), and the compatible condition $\partial_{tx} = \partial_{xt}$, one can show that the evolution equations for the curvature and torsion associated with a curve evolving under LLG are given by (see e.g. [DL83])

$$\begin{aligned} \partial_t \tau &= \beta\left(\mathrm{k}\partial_x\mathrm{k} + \partial_x\Big(\frac{\partial_{xx}\mathrm{k} - \mathrm{k}\tau^2}{\mathrm{k}}\Big)\right) + \alpha\left(\mathrm{k}^2\tau + \partial_x\Big(\frac{\partial_x(\mathrm{k}\tau) + \tau\partial_x\mathrm{k}}{\mathrm{k}}\Big)\right), \\ \partial_t \mathrm{k} &= \beta\left(-\partial_x(\mathrm{k}\tau) - \tau\partial_x\mathrm{k}\right) + \alpha\left(\partial_{xx}\mathrm{k} - \mathrm{k}\tau^2\right). \end{aligned} \tag{18}$$

In the absence of damping (i.e. $\alpha = 0$), system (18) reduces to the intrinsic equations associated with the LIA equation. In addition, in this case, Hasimoto [Has72] established a remarkable connection with a nonlinear cubic Schrödinger equation through the *filament function* defined by

$$u(s,t) = \mathrm{k}(s,t)e^{i\int_0^s \tau(\sigma,t)\,d\sigma}, \tag{19}$$

in terms of the curvature k and the torsion τ of the curve. The Hasimoto transform also allows us to establish the connection between the LLG equation with certain nonlinear Schrödinger equations. Precisely, it can be shown [DL83] that if $\boldsymbol{m}$ evolves under (15) and considering the filament function u defined by (19), then u solves the following nonlocal damped Schrödinger equation

$$i\partial_t u + (\beta - i\alpha)\partial_{xx}u + \frac{u}{2}\left(\beta|u|^2 + 2\alpha\int_0^x \mathrm{Im}(\overline{u}\partial_x u) - A(t)\right) = 0, \tag{20}$$

where

$$A(t) = \left(\beta\left(\mathrm{k}^2 + \frac{2(\partial_{xx}\mathrm{k} - \mathrm{k}\tau^2)}{\mathrm{k}}\right) + 2\alpha\left(\frac{\partial_x(\mathrm{k}\tau) + \partial_x\mathrm{k}\tau}{\mathrm{k}}\right)\right)(0,t).$$

Notice that if $\alpha = 0$, equation (20) is the completely integrable cubic Schrödinger equation.

Chapter 1

The Cauchy problem for the LL equation

Despite some serious efforts to establish a complete Cauchy theory for the LL equation, several issues remain unknown. In this chapter we will focus on the LL equation without damping, for which the Cauchy theory is even more delicate to handle. Even in the case where the problem is isotropic, i.e. the Schrödinger map equation, there are several unknowns aspects. Moreover, it is not always possible to adapt results for Schrödinger map equation to include anisotropic perturbations.

The aim of this chapter present results of [dLG15a, dLG18], on the Cauchy theory for smooth solutions to the anisotropic LL equation in Sobolev spaces. We will also provide the existence and uniqueness of a continuous flow in the energy space, in dimension $N = 1$.

The study of well-posedness in the presence of a damping term is different. Indeed, for the LLG equation, some techniques related to parabolic equations and for the heat-flow for harmonic maps (HFHM) can be used. We will discuss this issue in Chapter 4.

1.1 The Cauchy problem for smooth solutions

Let us consider the anisotropic LL equation

$$\partial_t \boldsymbol{m} + \boldsymbol{m} \times (\Delta \boldsymbol{m} - \lambda_1 m_1 \boldsymbol{e}_1 - \lambda_3 m_3 \boldsymbol{e}_3) = 0, \tag{1.1}$$

with $\lambda_1, \lambda_3 \geq 0$. Since the associated energy is given by

$$E_{\lambda_1,\lambda_3}(\boldsymbol{m}) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \boldsymbol{m}|^2 + \lambda_1 m_1^2 + \lambda_3 m_3^2), \tag{1.2}$$

the natural functional setting for solving this equation is the energy set

$$\mathcal{E}_{\lambda_1,\lambda_3}(\mathbb{R}^N) := \big\{ \boldsymbol{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^3) : |\boldsymbol{v}| = 1 \text{ a.e., } \nabla \boldsymbol{v} \in L^2(\mathbb{R}^N),\ \lambda_1 v_1, \lambda_3 v_3 \in L^2(\mathbb{R}^N) \big\}.$$

In the context of function taking values on $\mathbb{S}^2$, it is standard to use the notation

$$\mathcal{H}^\ell(\mathbb{R}^N) = \{\boldsymbol{v} \in L^1_{\rm loc}(\mathbb{R}^N, \mathbb{R}^3) : |\boldsymbol{v}| = 1 \text{ a.e., } \nabla \boldsymbol{v} \in H^{\ell-1}(\mathbb{R}^N)\},$$

for an integer $\ell \geq 1$. Notice that a function $\boldsymbol{v} \in \mathcal{H}^\ell(\mathbb{R}^N)$ does not belong to $L^2(\mathbb{R}^N, \mathbb{R}^3)$, since this is incompatible with the constraint $|\boldsymbol{v}| = 1$. In this manner, $\mathcal{E}_{\lambda_1,\lambda_3}(\mathbb{R}^N)$ reduces to $\mathcal{H}^1(\mathbb{R}^N)$ if $\lambda_1 = \lambda_3 = 0$.

For the sake of simplicity, in this section we drop the subscripts λ_1 and λ_3, and denote the energy by $E(\boldsymbol{m})$ and the space by $\mathcal{E}(\mathbb{R}^N)$, since the constants λ_1 and λ_3 are fixed.

The first results concerning the existence of weak solutions of (1.1) in the energy space where obtained by Zhou and Guo in the one-dimensional case $N = 1$ [ZG84], and by Sulem, Sulem and Bardos [SSB86] for $N \geq 1$. The approach followed in [ZG84] was to consider a parabolic regularization by adding the term $\varepsilon\Delta\boldsymbol{m}$ and letting $\varepsilon \to 0$ (see e.g. [GD08]), while the strategy in [SSB86] relied on finite difference approximations and a weak compactness argument. In both cases, no uniqueness was obtained. The proof in [SSB86] can be generalized to include the anisotropic perturbation in (1.1), leading to the existence of a global (weak) solution as follows.

Theorem 1.1 ([SSB86]). *For any $\boldsymbol{m}_0 \in \mathcal{E}(\mathbb{R}^N)$, there exists a global solution of* (1.1) *with $\boldsymbol{m} \in L^\infty(\mathbb{R}^+, \mathcal{E}(\mathbb{R}^N))$.*

The uniqueness of the solution in Theorem 1.1 not known. To our knowledge, the well-posedness of the Landau-Lifshitz equation for general initial data in $\mathcal{E}(\mathbb{R}^N)$ remains an open question.

Let us now discuss some results about smooth solutions in $\mathcal{H}^k(\mathbb{R}^N)$, $k \in \mathbb{N}$, in the isotropic case $\lambda_1 = \lambda_3 = 0$. For an initial data in $\boldsymbol{m}_0 \in \mathcal{H}^k(\mathbb{R}^N)$, Sulem, Sulem and Bardos [SSB86] proved the local existence and uniqueness of a solution $\boldsymbol{m} \in L^\infty([0,T), \mathcal{H}^k(\mathbb{R}^N))$, provided that $k > N/2 + 2$[1]. By using a parabolic approximation, Ding and Wang [DW98] proved the local existence in $L^\infty([0,T), \mathcal{H}^k(\mathbb{R}^N))$, provided that $k > N/2 + 1$. They also study the difference between two solutions, obtaining uniqueness provided that the solutions are of class C^3. Another approach was used by McMahagan [McG07], showing the existence as the limit of solutions to approximating wave problems, using parallel transport to compare two solutions and to conclude local existence and uniqueness in $L^\infty([0,T), \mathcal{H}^k(\mathbb{R}^N))$, for $k > N/2 + 1$.

When $N = 1$, these results provided the local existence and uniqueness at level $\mathcal{H}^k(\mathbb{R}^N)$, for $k \geq 2$. Moreover, in this case the solutions are global in time (see [RRS09, CSU00]).

Of course, there is a large amount of other works with interesting results about the (local and global) existence and uniqueness for the LL equation and other related equations, see e.g. [BIKT11, GD08, Mos05, KTV14, GS02, GGKT08, JS12, SW18] and the references therein. However, it is not straightforward to adapt these works to obtain local well-posednes results for smooth solutions to equation (1.1). For this reason, in the rest of this section we provide

[1]Actually, in [SSB86] they do not study of the difference between two solutions. It is only asserted that uniqueness followed from regularity, which it is not clear in this case; see also [JS12].

an alternative proof for local well-posedness by introducing high order energy quantities with better symmetrization properties.

To study the Cauchy problem of smooth solutions, given an integer $k \geq 1$, we introduce the set

$$\mathcal{E}^k(\mathbb{R}^N) := \mathcal{E}(\mathbb{R}^N) \cap \mathcal{H}^k(\mathbb{R}^N),$$

which we endow with the metric structure provided by the norm

$$\|\boldsymbol{v}\|_{Z^k} := \big(\|\nabla \boldsymbol{v}\|_{H^{k-1}}^2 + \|\boldsymbol{v}_2\|_{L^\infty}^2 + \lambda_1 \|\boldsymbol{v}_1\|_{L^2}^2 + \lambda_3 \|\boldsymbol{v}_3\|_{L^2}^2 \big)^{\frac{1}{2}},$$

of the vector space

$$Z^k(\mathbb{R}^N) := \big\{ \boldsymbol{v} \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^3) : \nabla \boldsymbol{v} \in H^{k-1}(\mathbb{R}^N),\ \boldsymbol{v}_2 \in L^\infty(\mathbb{R}^N)\ \lambda_1 \boldsymbol{v}_1, \lambda_3 \boldsymbol{v}_3 \in L^2(\mathbb{R}^N) \big\}. \tag{1.3}$$

Observe that the energy space $\mathcal{E}(\mathbb{R}^N)$ identifies with $\mathcal{E}^1(\mathbb{R}^N)$. The uniform control on the second component $\boldsymbol{v}_2$ in the Z^k-norm ensures that map $\| \cdot \|_{Z^k}$ is a norm. Of course, this uniform control is not the only possible choice of the metric structure. The main result of this section is following local well-posedness result.

Theorem 1.2 ([dLG18]). *Let $\lambda_1, \lambda_3 \geq 0$ and $k \in \mathbb{N}$, with $k > N/2 + 1$. For any initial condition $\boldsymbol{m}^0 \in \mathcal{E}^k(\mathbb{R}^N)$, there exists $T_{\max} > 0$ and a unique solution $\boldsymbol{m} : \mathbb{R}^N \times [0, T_{\max}) \to \mathbb{S}^2$ to the LL equation* (1.1)*, which satisfies the following statements.*

(i) *The solution $\boldsymbol{m}$ belongs to $L^\infty([0,T], \mathcal{E}^k(\mathbb{R}^N))$ and $\partial_t \boldsymbol{m} \in L^\infty([0,T], \mathcal{H}^{k-2}(\mathbb{R}^N))$, for all $T \in (0, T_{\max})$.*

(ii) *If the maximal time of existence $T_{\max}$ is finite, then*

$$\int_0^{T_{\max}} \|\nabla \boldsymbol{m}(\cdot, t)\|_{L^\infty}^2 \, dt = \infty. \tag{1.4}$$

(iii) *The flow map $\boldsymbol{m}^0 \mapsto \boldsymbol{m}$ is well-defined and locally Lipschitz continuous from $\mathcal{E}^k(\mathbb{R}^N)$ to $\mathcal{C}^0([0,T], \mathcal{E}^{k-1}(\mathbb{R}^N))$, for all $T \in (0, T_{\max})$.*

(iv) *When $\boldsymbol{m}^0 \in \mathcal{E}^\ell(\mathbb{R}^N)$, with $\ell > k$, the solution $\boldsymbol{m}$ lies in $L^\infty([0,T], \mathcal{E}^\ell(\mathbb{R}^N))$, with $\partial_t \boldsymbol{m} \in L^\infty([0,T], H^{\ell-2}(\mathbb{R}^N))$, for all $T \in (0, T_{\max})$.*

(v) *The energy* (1.2) *is conserved along the flow.*

Theorem 1.2 provides the local well-posedness of the LL equation in the set $\mathcal{E}^k(\mathbb{R}^N)$. This kind of statement is standard in the context of hyperbolic systems (see e.g. [Tay11, Theorem 1.2]). The critical regularity for the equation is given by the condition $k = N/2$, so that local well-posedness is expected when $k > N/2 + 1$. This assumption is used to control uniformly the gradient of the solutions by the Sobolev embedding theorem.

1.1.1 Ideas of the proof

The construction of the solution $\boldsymbol{m}$ in Theorem 1.2 is based on the strategy developed by Sulem, Sulem and Bardos [SSB86], relaying on a compactness argument and good energy estimates. The compactness argument requires the density of smooth functions in the sets $\mathcal{E}^k(\mathbb{R}^N)$. Recall that these sets are equal to $Z^k(\mathbb{R}^N, \mathbb{S}^2)$ for any integer $k \geq 1$, where the vector spaces $Z^k(\mathbb{R}^N)$ are defined in (1.3). In particular, the sets $\mathcal{E}^k(\mathbb{R}^N)$ are complete metric spaces for the distance corresponding to the Z^k-norm. Using this norm, we have

Lemma 1.3 ([dLG15a],[dLG18])**.** *Let $k \in \mathbb{N}$, with $k > N/2$. Given any function $\boldsymbol{m} \in \mathcal{E}^k(\mathbb{R}^N)$, there exists a sequence of smooth functions $\boldsymbol{m}_n \in \mathcal{E}(\mathbb{R}^N)$, with $\nabla \boldsymbol{m}_n \in H^\infty(\mathbb{R}^N)$, such that the differences $\boldsymbol{m}_n - \boldsymbol{m}$ are in $H^k(\mathbb{R}^N)$, and satisfy*

$$\boldsymbol{m}_n - \boldsymbol{m} \to 0 \quad \text{in } H^k(\mathbb{R}^N),$$

as $n \to \infty$. In particular, we have $\|\boldsymbol{m}_n - \boldsymbol{m}\|_{Z^k} \to 0$.

Remark 1.4. This density result is not necessarily true when $k \leq N/2$ (see e.g. [SU83, Section 4] for a discussion about this claim).

Concerning the energy estimates, a key observation is that a (smooth) solution to (1.1) satisfies the equation

$$\partial_{tt}\boldsymbol{m} + \Delta^2\boldsymbol{m} - (\lambda_1 + \lambda_3)\big(\Delta m_1 \boldsymbol{e}_1 + \Delta m_3 \boldsymbol{e}_3\big) + \lambda_1\lambda_3\big(m_1 e_1 + m_3 e_3\big) = F(\boldsymbol{m}), \tag{1.5}$$

where we have set

$$\begin{aligned}
F(\boldsymbol{m}) := & \sum_{1 \leq i,j \leq N} \Big(\partial_i\big(2\langle\partial_i\boldsymbol{m}, \partial_j\boldsymbol{m}\rangle_{\mathbb{R}^3}\partial_j\boldsymbol{m} - |\partial_j\boldsymbol{m}|^2\partial_i\boldsymbol{m}\big) - 2\partial_{ij}\big(\langle\partial_i\boldsymbol{m}, \partial_j\boldsymbol{m}\rangle_{\mathbb{R}^3}\boldsymbol{m}\big)\Big) \\
& + \lambda_1\Big(\operatorname{div}\big((m_3^2 - 2m_1^2)\nabla\boldsymbol{m} + (m_1\boldsymbol{m} - m_3^2 e_1 + m_1 m_3 e_3)\nabla m_1 + (m_1 m_3 e_1 - m_3\boldsymbol{m} - m_1^2 e_3)\nabla m_3\big) \\
& \qquad + \nabla m_1 \cdot \big(m_1\nabla\boldsymbol{m} - \boldsymbol{m}\nabla m_1\big) + \nabla m_3 \cdot \big(\boldsymbol{m}\nabla m_3 - m_3\nabla\boldsymbol{m}\big) + m_3|\nabla\boldsymbol{m}|^2 e_3 \\
& \qquad + \big(m_1\nabla m_3 - m_3\nabla m_1\big)\cdot\big(\nabla m_1 e_3 - \nabla m_3 e_1\big) + \lambda_1 m_1^2\big(m_1 e_1 - \boldsymbol{m}\big)\Big) \\
& + \lambda_3\Big(\operatorname{div}\big((m_1^2 - 2m_3^2)\nabla\boldsymbol{m} + (m_1 m_3 e_3 - m_1\boldsymbol{m} - m_3^2 e_1)\nabla m_1 + (m_3\boldsymbol{m} - m_1^2 e_3 + m_1 m_3 e_1)\nabla m_3\big) \\
& \qquad + \nabla m_3 \cdot \big(m_3\nabla\boldsymbol{m} - \boldsymbol{m}\nabla m_3\big) + \nabla m_1 \cdot \big(\boldsymbol{m}\nabla m_1 - m_1\nabla\boldsymbol{m}\big) + m_1|\nabla\boldsymbol{m}|^2 e_1 \\
& \qquad + \big(m_1\nabla m_3 - m_3\nabla m_1\big)\cdot\big(\nabla m_1 e_3 - \nabla m_3 e_1\big) + \lambda_3 m_3^2\big(m_3 e_3 - \boldsymbol{m}\big)\Big) \\
& + \lambda_1\lambda_3\Big((m_1^2 + m_3^2)\boldsymbol{m} + m_1^2 m_3 e_3 + m_3^2 m_1 e_1\Big).
\end{aligned}$$

In order to derive this expression, we have used the pointwise identities

$$\langle\boldsymbol{m}, \partial_i\boldsymbol{m}\rangle_{\mathbb{R}^3} = \langle\boldsymbol{m}, \partial_{ii}\boldsymbol{m}\rangle_{\mathbb{R}^3} + |\partial_i\boldsymbol{m}|^2 = \langle\boldsymbol{m}, \partial_{iij}\boldsymbol{m}\rangle_{\mathbb{R}^3} + 2\langle\partial_i\boldsymbol{m}, \partial_{ij}\boldsymbol{m}\rangle_{\mathbb{R}^3} + \langle\partial_j\boldsymbol{m}, \partial_{ii}\boldsymbol{m}\rangle_{\mathbb{R}^3} = 0,$$

which hold for any $1 \leq i, j \leq N$, due to the property that $\boldsymbol{m}$ is valued into the sphere $\mathbb{S}^2$.

In view of (1.5), we define the (pseudo)energy of order $k \geq 2$, as

$$E_{\mathrm{LL}}^k(t) := \frac{1}{2}\Big(\|\partial_t \boldsymbol{m}\|_{\dot{H}^{k-2}} + \|\boldsymbol{m}\|_{\dot{H}^k} + (\lambda_1+\lambda_3)(\|m_1\|_{\dot{H}^{k-1}} + \|m_3\|_{\dot{H}^{k-1}}) + \lambda_1\lambda_3(\|m_1\|_{\dot{H}^{k-2}} + \|m_3\|_{\dot{H}^{k-2}})\Big),$$

for any $t \in [0,T]$. This high order energy is an anisotropic version of the one used in [SSB86].

To get good energy estimates, we need to use Moser estimates (also called tame estimates) in Sobolev spaces (see e.g. [Mos66, Hör97]). Using these estimates and differentiating E_{LL}^k, we obtain the following energy estimates.

Proposition 1.5. *Let $\lambda_1, \lambda_3 \geq 0$ and $k \in \mathbb{N}$, with $k > 1 + N/2$. Assume that $\boldsymbol{m}$ is a solution to* (1.1)*, which lies in $\mathcal{C}^0([0,T], \mathcal{E}^{k+2}(\mathbb{R}^N))$, with $\partial_t \boldsymbol{m} \in \mathcal{C}^0([0,T], H^k(\mathbb{R}^N))$.*

(i) *The LL energy is well-defined and conserved along flow, that is*

$$E_{\mathrm{LL}}^1(t) := E_{\mathrm{LL}}\big(\boldsymbol{m}(\cdot,t)\big) = E_{\mathrm{LL}}^1(0),$$

for any $t \in [0,T]$.

(ii) *Given any integer $2 \leq \ell \leq k$, the energies E_{LL}^ℓ are of class $\mathcal{C}^1$ on $[0,T]$, and there exists a positive number C_k, depending only on k, such that their derivatives satisfy*

$$\big[E_{\mathrm{LL}}^\ell\big]'(t) \leq C_k\big(1 + \|m_1(t)\|_{L^\infty}^2 + \|m_3(t)\|_{L^\infty}^2 + \|\nabla \boldsymbol{m}(t)\|_{L^\infty}^2\big)\, \Sigma_{\mathrm{LL}}^\ell(t), \tag{1.6}$$

for any $t \in [0,T]$. Here, we have set $\Sigma_{\mathrm{LL}}^\ell := \sum_{j=1}^\ell E_{\mathrm{LL}}^j$.

We next discretize the equation by using a finite-difference scheme. The a priori bounds remain available in this discretized setting. We then apply standard weak compactness and local strong compactness results in order to construct local weak solutions, which satisfy statement (i) in Theorem 1.2. Applying the Gronwall lemma to the inequalities in (1.6) prevents a possible blow-up when the condition in (1.4) is not satisfied.

Finally, we establish uniqueness, as well as continuity with respect to the initial datum, by computing energy estimates for the difference of two solutions. More precisely, we show

Proposition 1.6. *Let $\lambda_1, \lambda_3 \geq 0$, and $k \in \mathbb{N}$, with $k > N/2 + 1$. Consider two solutions $\boldsymbol{m}$ and $\tilde{\boldsymbol{m}}$ to* (1.1)*, which lie in $\mathcal{C}^0([0,T], \mathcal{E}^{k+1}(\mathbb{R}^N))$, with $\partial_t \boldsymbol{m}, \partial_t \tilde{\boldsymbol{m}} \in \mathcal{C}^0([0,T], H^{k-1}(\mathbb{R}^N))$, and set $\boldsymbol{u} := \tilde{\boldsymbol{m}} - \boldsymbol{m}$ and $\boldsymbol{v} := (\tilde{\boldsymbol{m}} + \boldsymbol{m})/2$.*

(i) *The function*

$$\mathfrak{E}_{\mathrm{LL}}^0(t) := \frac{1}{2}\int_{\mathbb{R}^N} |\boldsymbol{u}(x,t) - u_2(x,0)\boldsymbol{e}_2|^2\, dx,$$

is of class $\mathcal{C}^1$ on $[0,T]$, and there exists a positive number C such that

$$\big[\mathfrak{E}_{\mathrm{LL}}^0\big]'(t) \leq C\big(1 + \|\nabla \tilde{\boldsymbol{m}}(t)\|_{L^2} + \|\nabla \boldsymbol{m}(t)\|_{L^2} + \|\tilde{m}_1(t)\|_{L^2} + \|m_1(t)\|_{L^2} + \|\tilde{m}_3(t)\|_{L^2} + \|m_3(t)\|_{L^2}\big)\, \big(\|\boldsymbol{u}(t) - u_2^0 e_2\|_{L^2}^2 + \|\boldsymbol{u}(t)\|_{L^\infty}^2 + \|\nabla \boldsymbol{u}(t)\|_{L^2}^2 + \|\nabla u_2^0\|_{L^2}^2\big),$$

for any $t \in [0,T]$.

(ii) *The function*

$$\mathfrak{E}^1_{\rm LL}(t) := \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}\times\nabla\boldsymbol{v} + \boldsymbol{v}\times\nabla\boldsymbol{u}|^2\big)\, dx,$$

is of class $\mathcal{C}^1$ on $[0,T]$, and there exists a positive number C such that

$$\begin{aligned}\big[\mathfrak{E}^1_{\rm LL}\big]'(t) \leq C\big(1 + \|\nabla \boldsymbol{m}(t)\|_{L^\infty}^2 + \|\nabla \tilde{\boldsymbol{m}}(t)\|_{L^\infty}^2\big)\, \big(\|\boldsymbol{u}(t)\|_{L^\infty}^2 + \|\nabla \boldsymbol{u}(t)\|_{L^2}^2\big)\times \\ \times\big(1 + \|\nabla \boldsymbol{m}(t)\|_{L^\infty} + \|\nabla \tilde{\boldsymbol{m}}(t)\|_{L^\infty} + \|\nabla \boldsymbol{m}(t)\|_{H^1} + \|\nabla \tilde{\boldsymbol{m}}(t)\|_{H^1}\big).\end{aligned}$$

(iii) *Let $2 \leq \ell \leq k-1$. The function*

$$\begin{aligned}\mathfrak{E}^\ell_{\rm LL}(t) :=& \frac{1}{2}\Big(\|\partial_t \boldsymbol{u}\|_{\dot{H}^{k-2}} + \|\boldsymbol{u}\|_{\dot{H}^k} + (\lambda_1+\lambda_3)(\|u_1\|_{\dot{H}^{k-1}} + \|u_3\|_{\dot{H}^{k-1}}) \\ &+ \lambda_1\lambda_3(\|u_1\|_{\dot{H}^{k-2}} + \|u_3\|_{\dot{H}^{k-2}})\Big),\end{aligned}$$

is of class $\mathcal{C}^1$ on $[0,T]$, and there exists a positive number C_k, depending only on k, such that

$$\begin{aligned}\big[\mathfrak{E}^\ell_{\rm LL}\big]'(t) \leq C_k\Big(&1 + \|\nabla \boldsymbol{m}(t)\|_{H^\ell}^2 + \|\nabla \tilde{\boldsymbol{m}}(t)\|_{H^\ell}^2 + \|\nabla \boldsymbol{m}(t)\|_{L^\infty}^2 + \|\nabla \tilde{\boldsymbol{m}}(t)\|_{L^\infty}^2 \\ &+ \delta_{\ell=2}\big(\|\tilde{m}_1(t)\|_{L^2} + \|m_1(t)\|_{L^2} + \|\tilde{m}_3(t)\|_{L^2} + \|m_3(t)\|_{L^2}\big)\Big)\,\big(\mathfrak{S}^\ell_{\rm LL}(t) + \|\boldsymbol{u}(t)\|_{L^\infty}^2\big).\end{aligned}$$

Here, we have set $\mathfrak{S}^\ell_{\rm LL} := \sum_{j=0}^\ell \mathfrak{E}^j_{\rm LL}$.

When $\ell \geq 2$, the quantities $\mathfrak{E}^\ell_{\rm LL}$ in Proposition 1.6 are anisotropic versions of the ones used in [SSB86] for similar purposes. Their explicit form is related to the linear part of the second-order equation in (1.5). The quantity $\mathfrak{E}^0_{\rm LL}$ is tailored to close off the estimates.

The introduction of the quantity $\mathfrak{E}^1_{\rm LL}$ is of a different nature. The functions $\nabla \boldsymbol{u}$ and $\boldsymbol{u}\times\nabla\boldsymbol{v} + \boldsymbol{v}\times\nabla\boldsymbol{u}$ in its definition appear as the good variables for performing hyperbolic estimates at an H^1-level. They provide a better symmetrization corresponding to a further cancellation of the higher order terms. Without any use of the Hasimoto transform, or of parallel transport, this makes possible a direct proof of local well-posedness at an H^k-level, with $k > N/2+1$ instead of $k > N/2+2$.

1.1.2 Local well-posedness for smooth solutions to the HLL equation

In the following chapters we will also need the hydrodynamical version of the LL equation. As explained in §2, this change of variables is a reminiscent of the use of the Madelung transform [Mad26]. Indeed, assuming that the map $\check{m} := m_1 + im_2$, associated with a solution $\boldsymbol{m}$ to (1.1), does not vanish, we can write

$$\check{m} = (1-m_3^2)^{\frac{1}{2}}\big(\sin(\phi) + i\cos(\phi)\big).$$

Thus, setting the hydrodynamical variables $u := m_3$ and ϕ, we get the system

$$\begin{cases} \partial_t u = \operatorname{div}\big((1-u^2)\nabla\phi\big) - \dfrac{\lambda_1}{2}(1-u^2)\sin(2\phi), \\ \partial_t \phi = -\operatorname{div}\Big(\dfrac{\nabla u}{1-u^2}\Big) + u\dfrac{|\nabla u|^2}{(1-u^2)^2} - u|\nabla\phi|^2 + u\big(\lambda_3 - \lambda_1 \sin^2(\phi)\big), \end{cases} \tag{HLL}$$

at least as

$$|u| < 1 \quad \text{on } \mathbb{R}^N. \tag{1.7}$$

In order to analyze rigorously this regime, we introduce a functional setting in which we can legitimate the use of the hydrodynamical framework. Under the condition (1.7), it is natural to work in the Hamiltonian framework in which the solutions $\boldsymbol{m}$ have finite LL energy. In the hydrodynamical formulation, the LL energy is given by

$$E_{\mathrm{LL}}(u,\varphi) := \frac{1}{2}\int_{\mathbb{R}^N} \Big(\frac{|\nabla u|^2}{1-u^2} + (1-u^2)|\nabla\varphi|^2 + \lambda_1(1-u^2)\sin^2(\varphi) + \lambda_3 u^2\Big). \tag{1.8}$$

As a consequence of this formula, we work in the nonvanishing set

$$\mathcal{NV}_{\sin}(\mathbb{R}^N) := \big\{(u,\varphi) \in H^1(\mathbb{R}^N) \times H^1_{\sin}(\mathbb{R}^N) : |u| < 1 \text{ on } \mathbb{R}^N\big\},$$

where

$$H^1_{\sin}(\mathbb{R}^N) := \big\{v \in L^1_{\mathrm{loc}}(\mathbb{R}^N) : \nabla v \in L^2(\mathbb{R}^N),\ \sin(v) \in L^2(\mathbb{R}^N)\big\}.$$

The set $H^1_{\sin}(\mathbb{R}^N)$ is an additive group, which is naturally endowed with the pseudometric distance

$$d^1_{\sin}(v_1,v_2) := \Big(\big\|\sin(v_1-v_2)\big\|^2_{L^2} + \big\|\nabla v_1 - \nabla v_2\big\|^2_{L^2}\Big)^{\frac{1}{2}},$$

that vanishes if and only if $v_1 - v_2 \in \pi\mathbb{Z}$. This quantity is not a distance on the group $H^1_{\sin}(\mathbb{R}^N)$, but it is on the quotient group $H^1_{\sin}(\mathbb{R}^N)/\pi\mathbb{Z}$. In the sequel, we identify the set $H^1_{\sin}(\mathbb{R}^N)$ with this quotient group when necessary, in particular when a metric structure is required. This identification is not a difficulty as far as we deal with the hydrodynamical form of the LL equation and with the Sine–Gordon equation. Both the equations are indeed left invariant by adding a constant number in $\pi\mathbb{Z}$ to the phase functions. This property is one of the motivations for introducing the pseudometric distance $d^1_{\sin}$. We refer to [dLG18] for more details concerning this distance, as well as the set $H^1_{\sin}(\mathbb{R}^N)$.

To translate the analysis of the Cauchy problem for the LL equation into the hydrodynamical framework, we set for $k \geq 1$,

$$\mathcal{NV}^k_{\sin}(\mathbb{R}^N) := \big\{(u,\varphi) \in H^k(\mathbb{R}^N) \times H^k_{\sin}(\mathbb{R}^N) : |u| < 1 \text{ on } \mathbb{R}^N\big\}.$$

Here, the additive group $H^k_{\sin}(\mathbb{R}^N)$ is defined as

$$H^k_{\sin}(\mathbb{R}^N) := \big\{v \in L^1_{\mathrm{loc}}(\mathbb{R}^N) : \nabla v \in H^{k-1}(\mathbb{R}^N) \text{ and } \sin(v) \in L^2(\mathbb{R}^N)\big\}.$$

As before, we identify this group, when necessary, with the quotient group $H^k_{\sin}(\mathbb{R}^N)/\pi\mathbb{Z}$, and then we endow it with the distance

$$d^k_{\sin}(v_1, v_2) := \Big(\| \sin(v_1 - v_2)\|^2_{L^2} + \|\nabla v_1 - \nabla v_2\|^2_{H^{k-1}} \Big)^{\frac{1}{2}}.$$

With this notation at hand, the vanishing set $\mathcal{NV}_{\sin}(\mathbb{R}^N)$ identifies with $\mathcal{NV}^1_{\sin}(\mathbb{R}^N)$.

From Theorem 1.2, we obtain the following local well-posedness result for (HLL).

Corollary 1.7 ([dLG18]). *Let $\lambda_1, \lambda_3 \geq 0$, and $k \in \mathbb{N}$, with $k > N/2 + 1$. Given any pair $(u^0, \phi^0) \in \mathcal{NV}^k_{\sin}(\mathbb{R}^N)$, there exist a positive number $T_{\max}$ and a unique solution $(u, \phi) : \mathbb{R}^N \times [0, T_{\max}) \to (-1, 1) \times \mathbb{R}$ to* (HLL) *with initial data (u^0, ϕ^0), which satisfies the following statements.*

(i) The solution (u, ϕ) is in $L^\infty([0, T], \mathcal{NV}^k_{\sin}(\mathbb{R}^N))$, while $(\partial_t u, \partial_t \phi)$ is in $L^\infty([0, T], H^{k-2}(\mathbb{R}^N)^2)$, for any $0 < T < T_{\max}$.

(ii) If the maximal time of existence $T_{\max}$ is finite, then

$$\int_0^{T_{\max}} \Big(\Big\| \frac{\nabla u(\cdot, t)}{(1 - u(\cdot, t)^2)^{\frac{1}{2}}} \Big\|^2_{L^\infty} + \Big\| (1 - u(\cdot, t)^2)^{\frac{1}{2}} \nabla \phi(\cdot, t) \Big\|^2_{L^\infty} \Big) \, dt = \infty, \quad \text{or} \quad \lim_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty} = 1.$$

(iii) The flow map $(u^0, \phi^0) \mapsto (u, \phi)$ is well-defined, and locally Lipschitz continuous from $\mathcal{NV}^k_{\sin}(\mathbb{R}^N)$ to $\mathcal{C}^0([0, T], \mathcal{NV}^{k-1}_{\sin}(\mathbb{R}^N))$ for any $0 < T < T_{\max}$.

(iv) When $(u^0, \phi^0) \in \mathcal{NV}^\ell_{\sin}(\mathbb{R}^N)$, with $\ell > k$, the solution (u, ϕ) lies in $L^\infty([0, T], \mathcal{NV}^\ell_{\sin}(\mathbb{R}^N))$, with $(\partial_t u, \partial_t \phi) \in L^\infty([0, T], H^{\ell-2}(\mathbb{R}^N)^2)$ for any $0 < T < T_{\max}$.

(v) The LL energy in (1.8) *is conserved along the flow.*

Remark 1.8. Here as in the sequel, the set $L^\infty([0, T], H^k_{\sin}(\mathbb{R}^N))$ is defined as

$$L^\infty\big([0, T], H^k_{\sin}(\mathbb{R}^N)\big) = \big\{ v \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times [0, T], \mathbb{R}) : \sup_{0 \leq t \leq T} \| \sin(v(\cdot, t))\|_{L^2} + \|\nabla v(\cdot, t)\|_{H^{k-1}} < \infty \big\},$$

for any integer $k \geq 1$ and any positive number T. This definition is consistent with the fact that a family $(v(\cdot, t))_{0 \leq t \leq T}$ of functions in $H^k_{\sin}(\mathbb{R}^N)$ (identified with the quotient group $H^k_{\sin}(\mathbb{R}^N)/\pi\mathbb{Z}$) is then bounded with respect to the distance $d^k_{\sin}$. In particular, the set $L^\infty([0, T], \mathcal{NV}^k_{\sin}(\mathbb{R}^N))$ is given by

$$L^\infty\big([0, T], \mathcal{NV}^k_{\sin}(\mathbb{R}^N)\big) := \Big\{ (u, \phi) \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times [0, T], \mathbb{R}^2) : |u| < 1 \text{ on } \mathbb{R}^N \times [0, T]$$
$$\text{and} \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{H^{k-1}} + \| \sin(\phi(\cdot, t))\|_{L^2} + \|\nabla \phi(\cdot, t)\|_{H^{k-1}} < \infty \Big\}.$$

The proof of Corollary 1.7 is complicated by the metric structure corresponding to the set $H^k_{\sin}(\mathbb{R}^N)$. Establishing the continuity of the flow map with respect to the pseudometric distance $d^k_{\sin}$ is not so immediate. We by-pass this difficulty by using some simple trigonometric identities.

1.2 Local well-posedness in the energy space in dimension one

In this section we focus on the LL equation with easy-plane anisotropy in dimension one, that is

$$\partial_t \boldsymbol{m} + \boldsymbol{m} \times (\partial_{xx}\boldsymbol{m} - \lambda_3 m_3 \boldsymbol{e}_3) = 0. \tag{1.9}$$

As mentioned before, the isotropic case $\lambda_3 = 0$, the best result concerning the local and global well-posedness for initial data in $\mathcal{H}^2(\mathbb{R})$ [CSU00, NSVZ07, RRS09]. Theorem 1.2 gives us for instance, the $\mathcal{H}^2$-local well-posedness, while Theorem 1.1 provides the existence of a solution in $\mathcal{H}^1(\mathbb{R})$, i.e. in the energy space for the isotropic equation. The isotropic equation is energy critical in $\mathcal{H}^{1/2}$, so that one could think that local well-posedness at the $\mathcal{H}^1$-level would be simple to establish. In this direction, when the domain is the torus, some progress has been made at the $H^{3/2^+}$-level [CET15], and an ill-posedness type result is given in [JS12] for the $H^{1/2}$-weak topology.

The purpose of this section is to provide a local well-posedness theory for (1.9) in the energy space, in the case $\lambda_3 \geq 0$. To this end, we will use the hydrodynamical version of the equation, considering hydrodynamical variables $u := m_3$ and $w := -\partial_x \varphi$, which gives us after some simplifications, the system

$$\begin{cases} \partial_t u = \partial_x\big((u^2-1)w\big), \\ \partial_t w = \partial_x\Big(\dfrac{\partial_{xx}u}{1-u^2} + u\dfrac{(\partial_x u)^2}{(1-u^2)^2} + u\big(w^2-\lambda_3\big)\Big). \end{cases} \tag{HLL1d}$$

We introduce the notation $\mathfrak{u} := (u, w)$, that we will refer to as hydrodynamical pair. Notice that the LL energy is now expressed as

$$E(\mathfrak{u}) := \int_{\mathbb{R}} e(\mathfrak{u}) := \frac{1}{2}\int_{\mathbb{R}} \Big(\frac{(u')^2}{1-u^2} + \big(1-u^2\big)w^2 + \lambda_3 u^2\Big). \tag{1.10}$$

Another formally conserved quantity is the momentum P, which is defined by

$$P(\mathfrak{u}) := \int_{\mathbb{R}} uw.$$

As we will see in Chapter 3, the momentum P, as well as the energy E, play an important role in the construction and the qualitative analysis of the solitons.

In order to establish this property rigorously, we first address the Cauchy problem in the hydrodynamical framework. In view of the expression of the energy in (1.10), the natural space for solving it is given by nonvanishing space

$$\mathcal{NV}(\mathbb{R}) := \Big\{\mathfrak{v} = (v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ s.t. } \max_{\mathbb{R}} |v| < 1\Big\},$$

endowed with the metric structure corresponding to the norm

$$\|\mathfrak{v}\|_{H^1\times L^2} := \Big(\|v\|_{H^1}^2 + \|w\|_{L^2}^2\Big)^{\frac{1}{2}}.$$

The nonvanishing condition on the maximum of $|v|$ is necessary to define properly the function w, which, in the original setting of a solution $\boldsymbol{m}$ to (1.9), corresponds to the derivative of the phase φ of the map $\check{m}$. Due to the Sobolev embedding theorem, this nonvanishing condition is also enough to define w properly, and then establish the continuity of the energy E and the momentum P on $\mathcal{NV}(\mathbb{R})$.

Concerning the Cauchy problem for (HLL1d), we have the following local well-posedness result.

Theorem 1.9 ([dLG15a])**.** *Let $\lambda_3 \geq 0$ and $\mathfrak{u}^0 = (u^0, w^0) \in \mathcal{NV}(\mathbb{R})$. There exist $T_{\max} > 0$ and $\mathfrak{u} = (u, w) \in \mathcal{C}^0([0, T_{\max}), \mathcal{NV}(\mathbb{R}))$, such that the following statements hold.*

(i) *The map $\mathfrak{u}$ is the unique solution to* (HLL1d)*, with initial condition $\mathfrak{u}^0$, such that there exist smooth solutions $\mathfrak{u}_n \in \mathcal{C}^\infty(\mathbb{R} \times [0, T])$ to* (HLL1d)*, which satisfy*

$$\mathfrak{u}_n \to \mathfrak{u} \quad \text{in } \mathcal{C}^0([0, T], \mathcal{NV}(\mathbb{R})), \tag{1.11}$$

as $n \to \infty$, for any $T \in (0, T_{\max})$.

(ii) *The maximal time $T_{\max}$ is characterized by the condition*

$$\lim_{t \to T_{\max}} \max_{x \in \mathbb{R}} |u(x, t)| = 1, \quad \text{if } T_{\max} < \infty.$$

(iii) *The energy E and the momentum P are constant along the flow.*

(iv) *When*

$$\mathfrak{u}_n^0 \to \mathfrak{u}^0 \quad \text{in } H^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

as $n \to \infty$, the maximal time of existence T_n of the solution $\mathfrak{u}_n$ to (HLL1d)*, with initial condition $\mathfrak{u}_n^0$, satisfies*

$$T_{\max} \leq \liminf_{n \to \infty} T_n,$$

and

$$\mathfrak{u}_n \to \mathfrak{u} \quad \text{in } \mathcal{C}^0([0, T], H^1(\mathbb{R}) \times L^2(\mathbb{R})),$$

as $n \to \infty$, for any $T \in (0, T_{\max})$.

In other words, Theorem 1.9 provides the existence and uniqueness of a continuous flow for (HLL1d) in the energy space $\mathcal{NV}(\mathbb{R})$. On the other hand, this does not prevent from the existence of other solutions which could not be approached by smooth solutions according to (1.11). In particular, we do not claim that there exists a unique local solution to (HLL1d) in the energy space for a given initial condition. To our knowledge, the question of the global existence in the hydrodynamical framework of the local solution $\mathfrak{v}$ remains open.

Concerning the equation (1.9), since we are in the one-dimensional case, we can endow the energy space

$$\mathcal{E}(\mathbb{R}) = \big\{ \boldsymbol{v} : \mathbb{R} \to \mathbb{S}^2 : \ \boldsymbol{v}' \in L^2(\mathbb{R}), \quad \lambda_3 v_3 \in L^2(\mathbb{R}) \big\},$$

with the metric structure corresponding to the distance

$$d_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{v}) := \Big(\big|\check{\boldsymbol{u}}(0) - \check{\boldsymbol{v}}(0)\big|^2 + \big\|\boldsymbol{u}' - \boldsymbol{v}'\big\|_{L^2}^2 + \lambda_3 \big\|u_3 - v_3\big\|_{L^2}^2 \Big)^{\frac{1}{2}}.$$

In this manner, we have the following statement for the original LL equation.

Corollary 1.10. *Let* $\lambda_3 \geq 0$ *and* $\boldsymbol{m}^0 \in \mathcal{E}(\mathbb{R})$*, with* $\max_{\mathbb{R}} |m_3^0| < 1$*. Consider the corresponding hydrodynamical pair* $\mathfrak{u}^0 \in \mathcal{NV}(\mathbb{R})$*, and denote by* $T_{\max} > 0$ *the maximal time of existence of the solution* $\mathfrak{u} \in \mathcal{C}^0([0, T_{\max}), \mathcal{NV}(\mathbb{R}))$ *to* (HLL1d) *with initial condition* $\mathfrak{u}^0$*, provided by Theorem 1.9. Then there exists* $\boldsymbol{m} \in \mathcal{C}^0([0, T_{\max}), \mathcal{E}(\mathbb{R}))$*, satisfying the following statements.*
(i) *The hydrodynamical pair corresponding to* $\boldsymbol{m}(x,t)$ *is well-defined for any* $(x,t) \in \mathbb{R} \times [0, T_{\max})$*, and is equal to* $\mathfrak{u}(x,t)$*.*
(ii) *The map* $\boldsymbol{m}$ *is the unique solution to* (1.9)*, with initial condition* $\boldsymbol{m}^0$*, such that there exist smooth solutions* $\boldsymbol{m}_n \in \mathcal{C}^\infty(\mathbb{R} \times [0,T])$ *to* (1.9)*, which satisfy*

$$\boldsymbol{m}_n \to \boldsymbol{m} \quad \text{in } \mathcal{C}^0([0,T], \mathcal{E}(\mathbb{R})),$$

as $n \to \infty$*, for any* $T \in (0, T_{\max})$*.*
(iii) *The energy* E *is constant along the flow.*
(iv) *If*

$$\boldsymbol{m}_n^0 \to \boldsymbol{m}^0 \quad \text{in } \mathcal{E}(\mathbb{R}),$$

as $n \to \infty$*, then the solution* $\boldsymbol{m}_n$ *to* (1.9) *with initial condition* $\boldsymbol{m}_n^0$ *satisfies*

$$\boldsymbol{m}_n \to \boldsymbol{m} \quad \text{in } \mathcal{C}^0([0,T], \mathcal{E}(\mathbb{R})),$$

as $n \to \infty$*, for any* $T \in (0, T_{\max})$*.*

Corollary 1.10 is nothing more than the translation of Theorem 1.9 in the original framework of the LL equation. It provides the existence of a unique continuous flow for (1.9) in the neighborhood of solutions $\boldsymbol{m}$, such that the third component $\boldsymbol{m}_3$ does not reach the value ± 1. The flow is only locally defined due to this restriction.

We refer to [dLG15a] for the detailed proofs of Theorem 1.9 and Corollary 1.10. Let us only explain a sketch of the proofs. The local well-posedness in the spaces $\mathcal{NV}^k(\mathbb{R})$ for the hydrodynamical system follows from the local well-posedness for the LL equation. The idea is that if $\boldsymbol{m}$ is a solution to (1.9), then

$$\mathfrak{u}(x,t) := \Big(m_3(x,t), \frac{m_1(x,t)\partial_x m_2(x,t) - m_2(x,t)\partial_x m_1(x,t)}{1 - m_3(x,t)^2} \Big),$$

solves (HLL1d). Reciprocally, let $\mathfrak{u} = (u, w)$ be a solution to (HLL1d). Considering φ a solution to the equation

$$\partial_t \varphi = \frac{1}{(1-u^2)^{\frac{1}{2}}} \partial_x \Big(\frac{\partial_x u}{(1-u^2)^{\frac{1}{2}}} \Big) + u\big(w^2 - 1\big),$$

we get that the map

$$\boldsymbol{m} := \Big((1-u^2)^{\frac{1}{2}}\cos(\varphi), (1-u^2)^{\frac{1}{2}}\sin(\varphi), u\Big),$$

solves (1.9). In this manner, from Theorem 1.2, we obtain

Proposition 1.11. *Let $k \geq 4$ and $\mathfrak{v}^0 = (u^0, w^0) \in \mathcal{NV}^k(\mathbb{R})$. There exists a positive maximal time $T_{\max}$, and a unique solution $\mathfrak{v} = (u, w)$ to* (HLL1d)*, with initial condition $\mathfrak{v}^0$, such that $\mathfrak{v}$ belongs to $\mathcal{C}^0([0, T_{\max}), \mathcal{NV}^{k-2}(\mathbb{R}))$, and $L^\infty([0, T], \mathcal{NV}^k(\mathbb{R}))$ for any $0 < T < T_{\max}$. The maximal time $T_{\max}$ is characterized by the condition*

$$\lim_{t \to T_{\max}} \|u(\cdot, t)\|_{\mathcal{C}^0} = 1, \quad \text{if} \quad T_{\max} < +\infty.$$

Moreover, the energy E and the momentum P are constant along the flow.

The most difficult part in Theorem 1.9 is the continuity with respect to the initial data in the energy space $\mathcal{NV}(\mathbb{R})$ when $\lambda_3 > 0$. In this case, by performing a change of variables, we can assume that $\lambda_3 = 1$. Our proof relies on the strategy developed by Chang, Shatah and Uhlenbeck in [CSU00] (see also [GS02, NSVZ07]). We introduce the map

$$\Psi := \frac{1}{2}\Big(\frac{\partial_x u}{(1-u^2)^{\frac{1}{2}}} + i(1-u^2)^{\frac{1}{2}} w\Big) \exp i\theta,$$

where we have set

$$\theta(x,t) := -\int_{-\infty}^x u(y,t) w(y,t)\, dy.$$

The map Ψ solves the nonlinear Schrödinger equation

$$i\partial_t \Psi + \partial_{xx}\Psi + 2|\Psi|^2\Psi + \frac{1}{2}u^2\Psi - \operatorname{Re}\Big(\Psi\big(1 - 2F(u, \overline{\Psi})\big)\Big)\big(1 - 2F(u, \Psi)\big) = 0, \tag{1.12}$$

with

$$F(u, \Psi)(x,t) := \int_{-\infty}^x u(y,t)\Psi(y,t)\, dy,$$

while the function u satisfies the two equations:

$$\begin{cases} \partial_t u = 2\partial_x \operatorname{Im}\Big(\Psi\big(2F(u, \overline{\Psi}) - 1\big)\Big), \\ \partial_x u = 2\operatorname{Re}\Big(\Psi\big(1 - 2F(u, \overline{\Psi})\big)\Big). \end{cases} \tag{1.13}$$

In this setting, deriving the continuous dependence in $\mathcal{NV}(\mathbb{R})$ of $\mathfrak{v}$ with respect to its initial data reduces to establish it for u and Ψ in $L^2(\mathbb{R})$. This is done in the following proposition, by combining an energy method for u and classical Strichartz estimates for Ψ.

Proposition 1.12. *Let $(v^0, \Psi^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and $(\tilde{v}^0, \tilde{\Psi}^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ be such that*

$$\partial_x v^0 = 2\operatorname{Re}\Big(\Psi^0\big(1 - 2F(v^0, \overline{\Psi^0})\big)\Big), \quad \text{and} \quad \partial_x \tilde{v}^0 = 2\operatorname{Re}\Big(\tilde{\Psi}^0\Big(1 - 2F\big(\tilde{v}^0, \overline{\tilde{\Psi}^0}\big)\Big)\Big).$$

Given two solutions (v, Ψ) *and* $(\tilde{v}, \tilde{\Psi})$ *in* $\mathcal{C}^0([0, T_*], H^1(\mathbb{R}) \times L^2(\mathbb{R}))$*, with* $(\Psi, \tilde{\Psi}) \in L^4([0, T_*], L^\infty(\mathbb{R}))^2$*, to* (1.12)-(1.13) *with initial datum* (v^0, Ψ^0)*, resp.* $(\tilde{v}^0, \tilde{\Psi}^0)$*, for some positive time* T_**, there exist a positive number* τ*, depending only on* $\|v^0\|_{L^2}$*,* $\|\tilde{v}^0\|_{L^2}$*,* $\|\Psi^0\|_{L^2}$ *and* $\|\tilde{\Psi}^0\|_{L^2}$*, and a universal constant* A *such that we have*

$$\big\|v - \tilde{v}\big\|_{\mathcal{C}^0([0,T],L^2)} + \big\|\Psi - \tilde{\Psi}\big\|_{\mathcal{C}^0([0,T],L^2)} + \big\|\Psi - \tilde{\Psi}\big\|_{L^4([0,T],L^\infty)} \leq A\Big(\big\|v^0 - \tilde{v}^0\big\|_{L^2} + \big\|\Psi^0 - \tilde{\Psi}^0\big\|_{L^2}\Big),$$

for any $T \in [0, \min\{\tau, T_*\}]$*. In addition, there exists a positive number* B*, depending only on* $\|v^0\|_{L^2}$*,* $\|\tilde{v}^0\|_{L^2}$*,* $\|\Psi^0\|_{L^2}$ *and* $\|\tilde{\Psi}^0\|_{L^2}$*, such that*

$$\big\|\partial_x v - \partial_x \tilde{v}\big\|_{\mathcal{C}^0([0,T],L^2)} \leq B\Big(\big\|v^0 - \tilde{v}^0\big\|_{L^2} + \big\|\Psi^0 - \tilde{\Psi}^0\big\|_{L^2}\Big),$$

for any $T \in [0, \min(\tau, T_*)]$.

Chapter 2

Asymptotic regimes for the anisotropic LL equation

In this chapter we will study the connection between the LL equation,

$$\partial_t \boldsymbol{m} + \boldsymbol{m} \times (\Delta \boldsymbol{m} - \lambda_1 m_1 \boldsymbol{e}_1 - \lambda_3 m_3 \boldsymbol{e}_3) = 0, \tag{2.1}$$

with $\lambda_1, \lambda_3 \geq 0$, and the Sine–Gordon and NLS equation, for certain types of anisotropies. More precisely, we consider the anisotropic LL equation (2.1) and investigate the cases when $\lambda_1 \ll \lambda_3$ and when $1 \ll \lambda_1 = \lambda_3$. A conjecture in the physical literature [Skl79, FT07] is that in the former case, the dynamics of (2.1) can be described by the Sine–Gordon equation, while in the latter case, can be approximated by the cubic NLS equation. Here we present a quantified proof in Sobolev norms of this conjecture, obtained in collaboration with P. Gravejat in [dLG18, dLG21].

It is well-known that deriving asymptotic regimes is a powerful tool in order to tackle the analysis of intricate equations. In this direction, we expect that the rigorous derivations in this chapter of the Sine–Gordon and the cubic Schrödinger regime will be a useful tool in order to describe the dynamical properties of the LL equation, in particular the role played by the solitons in this dynamics. For instance, this kind of strategy has been useful in order to prove the asymptotic stability of the dark solitons of the Gross-Pitaevskii equation by using its link with the Korteweg-de Vries equation (see [CR10, BGSS09, BGSS10]).

2.1 The Sine–Gordon regime

In order to provide a rigorous mathematical statement for the anisotropic LL equation with $\lambda_1 \ll \lambda_3$, i.e. for a strong easy-plane anisotropy regime, we consider a small parameter $\varepsilon > 0$, a fixed constant $\sigma > 0$, and set the anisotropy values

$$\lambda_1 := \sigma\varepsilon \qquad \text{and} \qquad \lambda_3 := \frac{1}{\varepsilon}.$$

We will also use the hydrodynamical formulation of the equation, i.e. assuming that the map $\check{m} := m_1 + im_2$, associated with a solution $\boldsymbol{m}$ to (2.1), does not vanish, it can be written as $\check{m} = (1-m_3^2)^{\frac{1}{2}}\big(\sin(\phi) + i\cos(\phi)\big)$. Thus, the hydrodynamical variables $u := m_3$ and ϕ satisfies the system

$$\begin{cases} \partial_t u = \operatorname{div}\big((1-u^2)\nabla\phi\big) - \dfrac{\sigma\varepsilon}{2}(1-u^2)\sin(2\phi), \\ \partial_t\phi = -\operatorname{div}\Big(\dfrac{\nabla u}{1-u^2}\Big) + u\dfrac{|\nabla u|^2}{(1-u^2)^2} - u|\nabla\phi|^2 + u\Big(\dfrac{1}{\varepsilon} - \sigma\varepsilon\sin^2(\phi)\Big), \end{cases}$$

as long as the nonvanishing condition (1.7) is satisfied. To study the behavior of the system as $\varepsilon \to 0$, we introduce the rescaled variables U_ε and Φ_ε given by

$$U_\varepsilon(x,t) = \frac{u(x/\sqrt{\varepsilon},t)}{\varepsilon}, \quad \text{and} \quad \Phi_\varepsilon(x,t) = \phi(x/\sqrt{\varepsilon},t),$$

which satisfy the hydrodynamical system

$$\begin{cases} \partial_t U_\varepsilon = \operatorname{div}\big((1-\varepsilon^2U_\varepsilon^2)\nabla\Phi_\varepsilon\big) - \dfrac{\sigma}{2}(1-\varepsilon^2U_\varepsilon^2)\sin(2\Phi_\varepsilon), \\ \partial_t\Phi_\varepsilon = U_\varepsilon\big(1-\varepsilon^2\sigma\sin^2(\Phi_\varepsilon)\big) - \varepsilon^2\operatorname{div}\Big(\dfrac{\nabla U_\varepsilon}{1-\varepsilon^2U_\varepsilon^2}\Big) + \varepsilon^4U_\varepsilon\dfrac{|\nabla U_\varepsilon|^2}{(1-\varepsilon^2U_\varepsilon^2)^2} - \varepsilon^2U_\varepsilon|\nabla\Phi_\varepsilon|^2. \end{cases} \tag{H_ε}$$

Therefore, as $\varepsilon \to 0$, we formally see that the limit system is

$$\begin{cases} \partial_t U = \Delta\Phi - \frac{\sigma}{2}\sin(2\Phi), \\ \partial_t\Phi = U, \end{cases} \tag{SGS}$$

so that the limit function Φ is a solution to the Sine–Gordon equation

$$\partial_{tt}\Phi - \Delta\Phi + \frac{\sigma}{2}\sin(2\Phi) = 0. \tag{SG}$$

As seen in Corollary 1.7 in Section 1.1.2, the hydrodynamical system (H_ε) is locally well-posed in the space $\mathcal{NV}^k_{\sin}(\mathbb{R}^N)$ for $k > N/2+1$, where

$$\mathcal{NV}^k_{\sin}(\mathbb{R}^N) := \big\{(u,\varphi) \in H^k(\mathbb{R}^N) \times H^k_{\sin}(\mathbb{R}^N) : |u| < 1 \text{ on } \mathbb{R}^N\big\}.$$

and

$$H^k_{\sin}(\mathbb{R}^N) := \big\{v \in L^1_{\mathrm{loc}}(\mathbb{R}^N) : \nabla v \in H^{k-1}(\mathbb{R}^N) \text{ and } \sin(v) \in L^2(\mathbb{R}^N)\big\}.$$

However, this result gives us time of existence T_ε that could vanish as $\varepsilon \to 0$. Therefore, we need to find a uniform estimate for T_ε to prevent this phenomenon. As we will discuss later, the Sine–Gordon equation is also locally well-posed at the same level of regularity, so that we can compare the evolution of the difference, at least in an interval of time independent of ε. A further analysis of (H_ε) involving good energy estimates, will lead us to the main result of this section, as follows.

Theorem 2.1 ([dLG18]). *Let $N \geq 1$ and $k \in \mathbb{N}$, with $k > N/2 + 1$, and $0 < \varepsilon < 1$. Consider an initial condition $(U_\varepsilon^0, \Phi_\varepsilon^0) \in \mathcal{NV}_{\sin}^{k+2}(\mathbb{R}^N)$, and set*

$$\mathcal{K}_\varepsilon := \big\|U_\varepsilon^0\big\|_{H^k} + \varepsilon\big\|\nabla U_\varepsilon^0\big\|_{H^k} + \big\|\nabla \Phi_\varepsilon^0\big\|_{H^k} + \big\|\sin(\Phi_\varepsilon^0)\big\|_{H^k}.$$

Consider similarly an initial condition $(U^0, \Phi^0) \in L^2(\mathbb{R}^N) \times H_{\sin}^1(\mathbb{R}^N)$, and denote by $(U, \Phi) \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}^N) \times H_{\sin}^1(\mathbb{R}^N))$ the unique corresponding solution to (SGS)*. Then, there exists $C > 0$, depending only on σ, k and N, such that, if the initial data satisfy the condition*

$$C\,\varepsilon\,\mathcal{K}_\varepsilon \leq 1, \tag{2.2}$$

then the following statements hold.

(i) There exists a positive number

$$T_\varepsilon \geq \frac{1}{C\mathcal{K}_\varepsilon^2}, \tag{2.3}$$

such that there is a unique solution $(U_\varepsilon, \Phi_\varepsilon) \in \mathcal{C}^0([0, T_\varepsilon], \mathcal{NV}_{\sin}^{k+1}(\mathbb{R}^N))$ to (H_ε) *with initial data $(U_\varepsilon^0, \Phi_\varepsilon^0)$.*

(ii) If $\Phi_\varepsilon^0 - \Phi^0 \in L^2(\mathbb{R}^N)$, then we have

$$\big\|\Phi_\varepsilon(\cdot, t) - \Phi(\cdot, t)\big\|_{L^2} \leq C\left(\big\|\Phi_\varepsilon^0 - \Phi^0\big\|_{L^2} + \big\|U_\varepsilon^0 - U^0\big\|_{L^2} + \varepsilon^2\,\mathcal{K}_\varepsilon\,\big(1 + \mathcal{K}_\varepsilon^3\big)\right)e^{Ct}, \tag{2.4}$$

for any $0 \leq t \leq T_\varepsilon$.

(iii) If $N \geq 2$, or $N = 1$ and $k > N/2 + 2$, then we have

$$\big\|U_\varepsilon(t) - U(t)\big\|_{L^2} + d_{\sin}^1(\Phi_\varepsilon(t) - \Phi(t)) \leq C\Big(\big\|U_\varepsilon^0 - U^0\big\|_{L^2} + d_{\sin}^1(\Phi_\varepsilon^0 - \Phi^0) + \varepsilon^2\,\mathcal{K}_\varepsilon\,\big(1 + \mathcal{K}_\varepsilon^3\big)\Big)e^{Ct}, \tag{2.5}$$

for any $0 \leq t \leq T_\varepsilon$.

(iv) Let $(U^0, \Phi^0) \in H^k(\mathbb{R}^N) \times H_{\sin}^{k+1}(\mathbb{R}^N)$ and set

$$\kappa_\varepsilon := \mathcal{K}_\varepsilon + \big\|U^0\big\|_{H^k} + \big\|\nabla\Phi^0\big\|_{H^k} + \big\|\sin(\Phi^0)\big\|_{H^k}.$$

There exists $A > 0$, depending only on σ, k and N, such that the solution (U, Φ) lies in $\mathcal{C}^0([0, T_\varepsilon^*], H^k(\mathbb{R}^N) \times H_{\sin}^{k+1}(\mathbb{R}^N))$, for a positive number

$$T_\varepsilon \geq T_\varepsilon^* \geq \frac{1}{A\kappa_\varepsilon^2}. \tag{2.6}$$

Moreover, when $k > N/2 + 3$, we have

$$\begin{aligned}&\big\|U_\varepsilon(\cdot, t) - U(\cdot, t)\big\|_{H^{k-3}} + \big\|\nabla\Phi_\varepsilon(\cdot, t) - \nabla\Phi(\cdot, t)\big\|_{H^{k-3}} + \big\|\sin(\Phi_\varepsilon(\cdot, t) - \Phi(\cdot, t))\big\|_{H^{k-3}}\\ &\leq A\,e^{A(1+\kappa_\varepsilon^2)t}\big(\big\|U_\varepsilon^0 - U^0\big\|_{H^{k-3}} + \big\|\nabla\Phi_\varepsilon^0 - \nabla\Phi^0\big\|_{H^{k-3}} + \big\|\sin(\Phi_\varepsilon^0 - \Phi^0)\big\|_{H^{k-3}} + \varepsilon^2\kappa_\varepsilon\big(1 + \kappa_\varepsilon^3\big)\big),\end{aligned} \tag{2.7}$$

for any $0 \leq t \leq T_\varepsilon^*$.

In arbitrary dimension, Theorem 2.1 provides a quantified convergence of the LL equation towards the Sine–Gordon equation in the regime of strong easy-plane anisotropy. Three types of convergence are proved depending on the dimension, and the levels of regularity of the solutions. This trichotomy is related to the analysis of the Cauchy problems for the LL and Sine–Gordon equations.

In its natural Hamiltonian framework, the Sine–Gordon equation is globally well-posed and its Hamiltonian is the Sine–Gordon energy:

$$E_{\mathrm{SG}}(\phi) = \frac{1}{2} \int_{\mathbb{R}^N} \big((\partial_t \phi)^2 + |\nabla \phi|^2 + \sigma \sin(\phi)^2 \big). \tag{2.8}$$

More precisely, given an initial condition $(\Phi^0, \Phi^1) \in H^1_{\mathrm{sin}}(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, there exists a unique corresponding solution $\Phi \in \mathcal{C}^0(\mathbb{R}, H^1_{\mathrm{sin}}(\mathbb{R}^N))$ to (SG), with $\partial_t \Phi \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}^N))$. Moreover, the Sine–Gordon equation is locally well-posed in the spaces $H^k_{\mathrm{sin}}(\mathbb{R}^N) \times H^{k-1}(\mathbb{R}^N)$, when $k > N/2 + 1$. In other words, the solution Φ remains in $\mathcal{C}^0([0, T], H^k_{\mathrm{sin}}(\mathbb{R}^N))$, with $\partial_t \Phi \in \mathcal{C}^0([0, T], H^{k-1}(\mathbb{R}^N))$, at least locally in time, when $(\Phi^0, \Phi^1) \in H^k_{\mathrm{sin}}(\mathbb{R}^N) \times H^{k-1}(\mathbb{R}^N)$. We will give more details about the Cauchy problem for (SG) in Subsection 2.1.1.

As seen in Chapter 1, the Cauchy problem for the LL equation at its Hamiltonian level is far from being completely understood. However, the LL equation is locally well-posed at the same level of high regularity as the Sine–Gordon equation. In the hydrodynamical context, this reads as the existence of a maximal time $T_{\max}$ and a unique solution $(U, \Phi) \in \mathcal{C}^0([0, T_{\max}), \mathcal{NV}^{k-1}_{\mathrm{sin}}(\mathbb{R}^N))$ to (H_ε) corresponding to an initial condition $(U^0, \Phi^0) \in \mathcal{NV}^k_{\mathrm{sin}}(\mathbb{R}^N)$, when $k > N/2 + 1$ (see Corollary 1.7 in Subsection 1.1); note the loss of one derivative here. This loss explains why we take initial conditions $(U^0_\varepsilon, \Phi^0_\varepsilon)$ in $\mathcal{NV}^{k+2}_{\mathrm{sin}}(\mathbb{R}^N)$, though the quantity $\mathcal{K}_\varepsilon$ is already well-defined when $(U^0_\varepsilon, \Phi^0_\varepsilon) \in \mathcal{NV}^{k+1}_{\mathrm{sin}}(\mathbb{R}^N)$.

In view of this local well-posedness result, we restrict our analysis of the Sine–Gordon regime to the solutions $(U_\varepsilon, \Phi_\varepsilon)$ to the rescaled system (H_ε) with sufficient regularity. A further difficulty then lies in the fact that their maximal times of existence possibly depend on the small parameter ε.

Statement (i) in Theorem 2.1 provides an explicit control on these maximal times. In view of (2.3), these maximal times are bounded from below by a positive number depending only on the choice of the initial data $(U^0_\varepsilon, \Phi^0_\varepsilon)$. Notice in particular that if a family of initial data $(U^0_\varepsilon, \Phi^0_\varepsilon)$ converges towards a pair (U^0, Φ^0) in $H^k(\mathbb{R}^N) \times H^k_{\mathrm{sin}}(\mathbb{R}^N)$, as $\varepsilon \to 0$, then it is possible to find $T > 0$ such that all the corresponding solutions $(U_\varepsilon, \Phi_\varepsilon)$ are well-defined on $[0, T]$. This property is necessary in order to make possible a consistent analysis of the limit $\varepsilon \to 0$.

Statement (i) only holds when the initial data $(U^0_\varepsilon, \Phi^0_\varepsilon)$ satisfy the condition in (2.2). However, this condition is not a restriction in the limit $\varepsilon \to 0$. It is satisfied by any fixed pair $(U^0, \Phi^0) \in \mathcal{NV}^{k+1}_{\mathrm{sin}}(\mathbb{R}^N)$ provided that ε is small enough, so that it is also satisfied by a family of initial data $(U^0_\varepsilon, \Phi^0_\varepsilon)$, which converges towards a pair (U^0, Φ^0) in $H^k(\mathbb{R}^N) \times H^k_{\mathrm{sin}}(\mathbb{R}^N)$ as $\varepsilon \to 0$.

Statements (ii) and (iii) in Theorem 2.1 provide two estimates between the previous solutions $(U_\varepsilon, \Phi_\varepsilon)$ to (H_ε), and an arbitrary global solution (U, Φ) to (SGS) at the Hamiltonian level. The first one yields an L^2-control on the difference $\Phi_\varepsilon - \Phi$, while the second one, an energetic control on the difference $(U_\varepsilon, \Phi_\varepsilon) - (U, \Phi)$. Due to the fact that the difference $\Phi_\varepsilon - \Phi$ is not necessarily in $L^2(\mathbb{R}^N)$, statement (ii) is restricted to initial conditions satisfying this property.

Finally, statement (iv) bounds the difference between the solutions $(U_\varepsilon, \Phi_\varepsilon)$ and (U, Φ) at the same initial Sobolev level. In this case, we also have to control the maximal time of regularity of the solutions (U, Φ). This follows from the control from below in (2.6), which is of the same order as the one in (2.3).

We then obtain the Sobolev estimate in (2.7) of the difference $(U_\varepsilon, \Phi_\varepsilon) - (U, \Phi)$ with a loss of three derivatives. Here, the choice of the Sobolev exponents $k > N/2 + 3$ is tailored to gain a uniform control on the functions $U_\varepsilon - U$, $\nabla\Phi_\varepsilon - \nabla\Phi$ and $\sin(\Phi_\varepsilon - \Phi)$, by the Sobolev embedding theorem.

A loss of derivatives is natural in the context of long-wave regimes (see e.g. [BGSS09, BGSS10] and the references therein). It is related to the terms with first and second-order derivatives in the right-hand side of (H_ε). This loss is the reason why the energetic estimate in statement (iii) requires an extra derivative in dimension one, that is the condition $k > N/2+2$. Using the Sobolev bounds (2.13) in Corollary 2.6 below, we can (partly) recover this loss by a standard interpolation argument, and deduce an estimate in $H^\ell(\mathbb{R}^N) \times H^{\ell+1}_{\sin}(\mathbb{R}^N)$ for any number $\ell < k$. In this case, the error terms are no more of order ε^2, as in the right-hand sides of (2.4), (2.5) and (2.7).

Our presentation of the convergence results in Theorem 2.1 is motivated by the fact that a control of order ε^2 is sharp. As a matter of fact, the system (SGS) owns explicit traveling-wave solutions. Up to a suitable scaling for which $\sigma = 1$, and up to the geometric invariance by translation, they are given by the kink and anti-kink functions

$$u_c^\pm(x,t) = \pm \frac{c}{\sqrt{1-c^2}\cosh\left(\frac{x-ct}{\sqrt{1-c^2}}\right)}, \quad \text{and} \quad \phi_c^\pm(x,t) = 2\arctan\left(e^{\mp\frac{x-ct}{\sqrt{1-c^2}}}\right), \tag{2.9}$$

for any speed $c \in (-1, 1)$. Using the explicit solitons in (10), we can get explicit traveling-wave solutions $(U_{c,\varepsilon}, \Phi_{c,\varepsilon})$ to (H_ε), with speed c, for which their exists a positive number A, depending only on c, such that

$$\|U_{c,\varepsilon} - u_c^+\|_{L^2} + \|\nabla\Phi_{c,\varepsilon} - \nabla\phi_c^+\|_{L^2} + \big\|\sin(\Phi_{c,\varepsilon} - \phi_c^+)\big\|_{L^2} \underset{\varepsilon\to 0}{\sim} A\varepsilon^2.$$

Hence, the estimates by ε^2 in (2.4), (2.5) and (2.7) are indeed optimal.

As a by-product of our analysis, we can also analyze the wave regime for the LL equation. This regime is obtained by allowing the parameter σ to converge to 0. At least formally, a solution $(U_{\varepsilon,\sigma}, \Phi_{\varepsilon,\sigma})$ to (H_ε) indeed satisfies the free wave system

$$\begin{cases} \partial_t U = \Delta\Phi, \\ \partial_t\Phi = U, \end{cases} \tag{FW}$$

as $\varepsilon \to 0$ and $\sigma \to 0$. In particular, the function Φ is solution to the free wave equation

$$\partial_{tt}\Phi - \Delta\Phi = 0.$$

The following result provides a rigorous justification for this asymptotic approximation.

Theorem 2.2 ([dLG18]). *Let $N \geq 1$ and $k \in \mathbb{N}$, with $k > N/2 + 1$, and $0 < \varepsilon, \sigma < 1$. Consider an initial condition $(U^0_{\varepsilon,\sigma}, \Phi^0_{\varepsilon,\sigma}) \in \mathcal{NV}^{k+2}_{\sin}(\mathbb{R}^N)$ and set*

$$\mathcal{K}^0_{\varepsilon,\sigma} := \big\|U^0_{\varepsilon,\sigma}\big\|_{H^k} + \varepsilon\big\|\nabla U^0_{\varepsilon,\sigma}\big\|_{H^k} + \big\|\nabla \Phi^0_{\varepsilon,\sigma}\big\|_{H^k} + \sigma^{\frac{1}{2}}\big\|\sin(\Phi^0_{\varepsilon,\sigma})\big\|_{L^2}.$$

Let $m \in \mathbb{N}$, with $0 \leq m \leq k-2$. Consider similarly an initial condition $(U^0, \Phi^0) \in H^m(\mathbb{R}^N) \times H^{m-1}(\mathbb{R}^N)$, and denote by $(U, \Phi) \in \mathcal{C}^0(\mathbb{R}, H^{m-1}(\mathbb{R}^N) \times H^m(\mathbb{R}^N))$ the unique corresponding solution to (FW)*. Then, there exists $C > 0$, depending only on k and N, such that, if the initial data satisfies the condition*

$$C\,\varepsilon\,\mathcal{K}^0_{\varepsilon,\sigma} \leq 1,$$

the following statements hold.

(i) There exists a positive number

$$T_{\varepsilon,\sigma} \geq \frac{1}{C\max(\varepsilon,\sigma)(1+\mathcal{K}^0_{\varepsilon,\sigma})^{\max(2,k/2)}},$$

such that there is a unique solution $(U_{\varepsilon,\sigma}, \Phi_{\varepsilon,\sigma}) \in \mathcal{C}^0([0, T_{\varepsilon,\sigma}], \mathcal{NV}^{k+1}_{\sin}(\mathbb{R}^N))$ to (H_ε) *with initial data $(U^0_{\varepsilon,\sigma}, \Phi^0_{\varepsilon,\sigma})$.*

(ii) If $\Phi^0_{\varepsilon,\sigma} - \Phi^0 \in H^m(\mathbb{R}^N)$, then we have the estimate

$$\begin{aligned}\big\|U_{\varepsilon,\sigma}(\cdot,t) - U(\cdot,t)\big\|_{H^{m-1}} + \big\|\Phi_{\varepsilon,\sigma}(\cdot,t) - \Phi(\cdot,t)\big\|_{H^m} \leq C\big(1+t^2\big)\Big(\big\|U^0_{\varepsilon,\sigma} - U^0\big\|_{H^{m-1}}\\ + \big\|\Phi^0_{\varepsilon,\sigma} - \Phi^0\big\|_{H^m} + \max\big(\varepsilon^2, \sigma^{1/2}\big)\,\mathcal{K}^0_{\varepsilon,\sigma}\,\big(1+\mathcal{K}^0_{\varepsilon,\sigma}\big)^{\max(2,m)}\Big),\end{aligned}$$

for any $0 \leq t \leq T_{\varepsilon,\sigma}$. In addition,

$$\begin{aligned}\big\|U_{\varepsilon,\sigma}(\cdot,t) - U(\cdot,t)\big\|_{\dot{H}^{\ell-1}} + \big\|\Phi_{\varepsilon,\sigma}(\cdot,t) - \Phi(\cdot,t)\big\|_{\dot{H}^\ell} \leq C\big(1+t\big)\Big(\big\|U^0_{\varepsilon,\sigma} - U^0\big\|_{\dot{H}^{\ell-1}}\\ + \big\|\Phi^0_{\varepsilon,\sigma} - \Phi^0\big\|_{\dot{H}^\ell} + \max\big(\varepsilon^2, \sigma\big)\,\mathcal{K}^0_{\varepsilon,\sigma}\big(1+\mathcal{K}^0_{\varepsilon,\sigma}\big)^{\max(2,\ell)}\Big),\end{aligned}$$

for any $1 \leq \ell \leq m$ and any $0 \leq t \leq T_{\varepsilon,\sigma}$.

The wave regime of the LL equation was first derived rigorously by Shatah and Zeng [SZ06], as a special case of the wave regimes for the Schrödinger map equations with values into arbitrary Kähler manifolds. The derivation in [SZ06] relies on energy estimates, which are similar in spirit to the ones we establish in the sequel, and a compactness argument. Getting rid of this compactness argument provides the quantified version of the convergence in Theorem 2.2. This improvement is based on the arguments developed by Béthuel, Danchin and Smets [BDS09] in order to quantify the convergence of the Gross–Pitaevskii equation towards the free wave equation in a similar long-wave regime. Similar arguments were also applied in [Chi14] in order to derive rigorously the (modified) Korteweg-de Vries and (modified) Kadomtsev–Petviashvili regimes of the LL equation (see also [GR19]).

In the remaining part of this section, we clarify the analysis of the Cauchy problem for the Sine–Gordon equation and detail the main ingredients in the proof of Theorem 2.1.

2.1.1 The Cauchy problem for the Sine–Gordon equation

The Sine–Gordon equation is a semilinear wave equation with a Lipschitz nonlinearity. The well-posedness analysis of the corresponding Cauchy problem is classical (see e.g. [SS98, Chapter 6] and [Eva10, Chapter 12]). With the proof of Theorem 2.1 in mind, we now provide some details about this analysis in the context of the product sets $H^k_{\rm sin}(\mathbb{R}^N) \times H^{k-1}(\mathbb{R}^N)$.

In the Hamiltonian framework, it is natural to solve the equation for initial conditions $\phi(\cdot, 0) = \phi^0 \in H^1_{\rm sin}(\mathbb{R}^N)$ and $\partial_t \phi(\cdot, 0) = \phi^1 \in L^2(\mathbb{R}^N)$, which guarantees the finiteness of the Sine–Gordon energy in (2.8). Note that we do not assume that the function ϕ^0 lies in $L^2(\mathbb{R}^N)$. This is motivated by formula (2.9) for the one-dimensional solitons $\phi_c^{\pm}$, which lie in $H^1_{\rm sin}(\mathbb{R})$, but not in $L^2(\mathbb{R})$. In this Hamiltonian setting, the Cauchy problem for (SG) is globally well-posed.

Theorem 2.3 ([dLG18])**.** *Let $\sigma \in \mathbb{R}^*$. Given two functions $(\phi^0, \phi^1) \in H^1_{\rm sin}(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, there exists a unique solution $\phi \in \mathcal{C}^0(\mathbb{R}, \phi^0 + H^1(\mathbb{R}^N))$, with $\partial_t \phi \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}^N))$, to the Sine–Gordon equation* (SG) *with initial conditions (ϕ^0, ϕ^1). Moreover, this solution satisfies the following properties.*

(i) For any $T > 0$, there exists $A > 0$, depending only on σ and T, such that the flow map $(\phi^0, \phi^1) \mapsto (\phi, \partial_t \phi)$ satisfies

$$d^1_{\rm sin}\big(\phi(\cdot, t), \tilde{\phi}(\cdot, t)\big) + \big\|\partial_t \phi(\cdot, t) - \partial_t \tilde{\phi}(\cdot, t)\big\|_{L^2} \leq A\Big(d^1_{\rm sin}\big(\phi^0, \tilde{\phi}^0\big) + \big\|\phi^1 - \tilde{\phi}^1\big\|_{L^2}\Big),$$

for any $t \in [-T, T]$. Here, the function $\tilde{\phi}$ is the unique solution to the Sine–Gordon equation with initial conditions $(\tilde{\phi}^0, \tilde{\phi}^1)$.

(ii) When $\phi^0 \in H^2_{\rm sin}(\mathbb{R}^N)$ and $\phi^1 \in H^1(\mathbb{R}^N)$, the solution ϕ belongs to the space $\mathcal{C}^0(\mathbb{R}, \phi^0 + H^2(\mathbb{R}^N))$, with $\partial_t \phi \in \mathcal{C}^0(\mathbb{R}, H^1(\mathbb{R}^N))$ and $\partial_{tt} \phi \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}^N))$.

(iii) The Sine–Gordon energy $E_{\rm SG}$ is conserved along the flow.

The proof of Theorem 2.3 relies on a classical fixed-point argument. The only difficulty consists in working in the unusual functional setting provided by the set $H^1_{\rm sin}(\mathbb{R}^N)$. This difficulty is by-passed by applying the strategy developed by Buckingham and Miller in [BM08, Appendix B] (see also [Gal08a, dL10] for similar arguments in the context of the Gross–Pitaevskii equation). In dimension $N = 1$, they fix a function $f \in \mathcal{C}^\infty(\mathbb{R})$, with (possibly different) limits $\ell^{\pm}\pi$ at $\pm\infty$, and with a derivative f' in the Schwartz class. Given a real number $p \geq 1$, they consider an initial data $(\phi^0 = f + \varphi^0, \phi^1)$, with $\varphi^0, \phi^1 \in L^p(\mathbb{R})$, and they apply a fixed-point argument in order to construct the unique corresponding solution $\phi = f + \varphi$ to the Sine–Gordon equation, with $\varphi \in L^\infty([0, T], L^p(\mathbb{R}))$ for some time $T > 0$. This solution is global when ϕ^0 lies in $W^{1,p}(\mathbb{R})$. This result includes all the functions ϕ^0 in the space $H^1_{\rm sin}(\mathbb{R})$ for $p = 2$.

Our proof of Theorem 2.3 extends this strategy to arbitrary dimensions. We fix a smooth function $f \in H^\infty_{\rm sin}(\mathbb{R}^N) := \cap_{k \geq 1} H^k_{\rm sin}(\mathbb{R}^N)$, and we apply a fixed-point argument in order to solve the Cauchy problem for initial conditions $\phi^0 \in f + H^1(\mathbb{R}^N)$ and $\phi^1 \in L^2(\mathbb{R}^N)$. We finally check the local Lipschitz continuity in $H^1_{\rm sin}(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ of the corresponding flow.

With the proof of Theorem 2.1 in mind, we also extend this analysis to the initial conditions $\phi^0 \in H^k_{\sin}(\mathbb{R}^N)$ and $\phi^1 \in H^{k-1}(\mathbb{R}^N)$, with $k \in \mathbb{N}^*$. When the integer k is large enough, we obtain the following local well-posedness result.

Theorem 2.4 ([dLG18]). *Let $\sigma \in \mathbb{R}^*$ and $k \in \mathbb{N}$, with $k > N/2 + 1$. Given two functions $(\phi^0, \phi^1) \in H^k_{\sin}(\mathbb{R}^N) \times H^{k-1}(\mathbb{R}^N)$, there exist $T^k_{\max} > 0$, and a unique solution $\phi \in \mathcal{C}^0([0, T^k_{\max}), \phi^0 + H^k(\mathbb{R}^N))$, with $\partial_t \phi \in \mathcal{C}^0([0, T^k_{\max}), H^{k-1}(\mathbb{R}^N))$, to the Sine–Gordon equation* (SG) *with initial conditions (ϕ^0, ϕ^1). Moreover, this solution satisfies the following properties.*

(i) The maximal time of existence $T^k_{\max}$ is characterized by the condition

$$\lim_{t \to T^k_{\max}} d^k_{\sin}\big(\phi(\cdot, t), 0\big) = \infty, \quad \text{if } T^k_{\max} < \infty.$$

(ii) Let $0 \leq T < T^k_{\max}$. There are $R > 0$ and $A > 0$, depending only on T, $d^k_{\sin}(\phi^0, 0)$ and $\|\phi^1\|_{H^{k-1}}$, such that the flow map $(\phi^0, \phi^1) \mapsto (\phi, \partial_t \phi)$ is well-defined from the ball

$$B\big((\phi^0, \phi^1), R\big) = \big\{(\tilde{\phi}^0, \tilde{\phi}^1) \in H^k_{\sin}(\mathbb{R}^N) \times H^{k-1}(\mathbb{R}^N) : d^k_{\sin}(\phi^0, \tilde{\phi}^0) + \|\phi^1 - \tilde{\phi}^1\|_{H^{k-1}} < R\big\},$$

to $\mathcal{C}^0([0, T], H^k_{\sin}(\mathbb{R}^N)) \times H^{k-1}(\mathbb{R}^N))$, and satisfies

$$d^k_{\sin}\big(\phi(\cdot, t), \tilde{\phi}(\cdot, t)\big) + \big\|\partial_t \phi(\cdot, t) - \partial_t \tilde{\phi}(\cdot, t)\big\|_{H^{k-1}} \leq A\Big(d^k_{\sin}\big(\phi^0, \tilde{\phi}^0\big) + \big\|\phi^1 - \tilde{\phi}^1\big\|_{H^{k-1}}\Big),$$

for any $t \in [0, T]$. Here, the function $\tilde{\phi}$ is the unique solution to the Sine–Gordon equation with initial conditions $(\tilde{\phi}^0, \tilde{\phi}^1)$.

(iii) When $\phi^0 \in H^{k+1}_{\sin}(\mathbb{R}^N)$ and $\phi^1 \in H^k(\mathbb{R}^N)$, the function ϕ is in $\mathcal{C}^0([0, T^k_{\max}), \phi^0 + H^{k+1}(\mathbb{R}^N))$, with $\partial_t \phi \in \mathcal{C}^0([0, T^k_{\max}), H^k(\mathbb{R}^N))$ and $\partial_{tt} \phi \in \mathcal{C}^0([0, T^k_{\max}), H^{k-1}(\mathbb{R}^N))$. In particular, the maximal time of existence $T^{k+1}_{\max}$ satisfies

$$T^{k+1}_{\max} = T^k_{\max}.$$

(iv) When $1 \leq N \leq 3$, the solution ϕ is global in time. Moreover, when $N \in \{2, 3\}$, the flow remains continuous for $k = 2$, i.e. on $H^2_{\sin}(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Theorem 2.4 follows from a fixed-point argument similar to the one of Theorem 2.3. The control on the nonlinear terms is derived from a uniform bound on the gradient of the solutions. This is the origin of the condition $k > N/2 + 1$ for which the Sobolev embedding theorem guarantees a uniform control on the gradient. This condition is natural in the context of the spaces $H^k_{\sin}(\mathbb{R}^N)$.

The maximal time of existence $T^k_{\max}$ in statement (i) can be estimated by performing standard energy estimates. When $1 \leq N \leq 3$, this leads to the global well-posedness of the Sine–Gordon equation in the space $H^k_{\sin}(\mathbb{R}^N) \times H^{k-1}(\mathbb{R}^N)$ for $k > N/2 + 1$. Actually, it is possible to extend this global well-posedness result to dimensions $4 \leq N \leq 9$. This extension relies on the Strichartz estimates for the free wave equation (see e.g. [KT98]), and the use of fractional Sobolev spaces. A blow-up in finite time is possible when $N \geq 10$. For the sake of simplicity, and since this is not our main goal, we do not address this question any further. We refer to [Tao16] for a detailed discussion on this topic, and for the construction of blowing-up solutions to related semilinear wave systems.

When $1 \leq N \leq 3$, the fixed-point arguments in the proofs of Theorems 2.3 and 2.4 provide the continuity of the flow with values in $\mathcal{C}^0([0,T], H^k_{\sin}(\mathbb{R}^N) \times H^{k-1}(\mathbb{R}^N))$ for any positive number T, except if $k = 2$ and $2 \leq N \leq 3$. We fill this gap by performing standard energy estimates. We conclude that the Sine–Gordon equation is globally well-posed in the spaces $H^k_{\sin}(\mathbb{R}^N) \times H^k(\mathbb{R}^N)$ for any $1 \leq N \leq 3$ and any $k \geq 1$.

Note finally that the previous well-posedness analysis of the Sine–Gordon equation translates immediately into the Sine–Gordon system (SGS) by setting $u = \partial_t \phi$.

2.1.2 Sketch of the proof of Theorem 2.1

When $(U^0_\varepsilon, \Phi^0_\varepsilon)$ lies in $\mathcal{NV}^{k+2}_{\sin}(\mathbb{R}^N)$, we deduce from Corollary 1.7 the existence of $T_{\max} > 0$, and of a unique solution $(U_\varepsilon, \Phi_\varepsilon) \in \mathcal{C}^0([0, T_{\max}), \mathcal{NV}^{k+1}_{\sin}(\mathbb{R}^N))$ to (H_ε) with initial data $(U^0_\varepsilon, \Phi^0_\varepsilon)$. The maximal time of existence $T_{\max}$ a priori depends on the scaling parameter ε. The number $T_{\max}$ might become smaller and smaller in the limit $\varepsilon \to 0$, so that analyzing this limit would have no sense.

As a consequence, our first task in the proof of Theorem 2.1 is to provide a control on $T_{\max}$. In view of the conditions in statement (ii) of Corollary 1.7, this control can be derived from uniform bounds on the functions U_ε, ∇U_ε and $\nabla \Phi_\varepsilon$. Taking into account the Sobolev embedding theorem and the fact that $k > N/2+1$, we are left with the computations of energy estimates for the functions U_ε and Φ_ε in the spaces $H^k(\mathbb{R}^N)$ and $H^k_{\sin}(\mathbb{R}^N)$, respectively.

In this direction, we recall that the LL energy corresponding to the scaled hydrodynamical system (H_ε) writes as

$$E_\varepsilon(U_\varepsilon, \Phi_\varepsilon) = \frac{1}{2} \int_{\mathbb{R}^N} \Big(\varepsilon^2 \frac{|\nabla U_\varepsilon|^2}{1 - \varepsilon^2 U_\varepsilon^2} + U_\varepsilon^2 + (1 - \varepsilon^2 U_\varepsilon^2) |\nabla \Phi_\varepsilon|^2 + \sigma (1 - \varepsilon^2 U_\varepsilon^2) \sin^2(\Phi_\varepsilon) \Big).$$

Inspired in this formula, we proposed to define an energy of order $k \geq 1$ as

$$\begin{aligned} E^k_\varepsilon(U_\varepsilon, \Phi_\varepsilon) = \frac{1}{2} \sum_{|\alpha| = k-1} \int_{\mathbb{R}^N} \Big(\varepsilon^2 \frac{|\nabla \partial^\alpha_x U_\varepsilon|^2}{1 - \varepsilon^2 U_\varepsilon^2} + |\partial^\alpha_x U_\varepsilon|^2 + (1 - \varepsilon^2 U_\varepsilon^2) |\nabla \partial^\alpha_x \Phi_\varepsilon|^2 \\ + \sigma (1 - \varepsilon^2 U_\varepsilon^2) |\partial^\alpha_x \sin(\Phi_\varepsilon)|^2 \Big). \end{aligned} \tag{2.10}$$

The factors $1 - \varepsilon^2 U_\varepsilon^2$ in this expression, as well as the non-quadratic term corresponding to the function $\sin(\Phi_\varepsilon)$, are of substantial importance since they provide a better symmetrization of the energy estimates, by inducing cancellations in the higher order terms. More precisely, we have

Proposition 2.5. *Let $\varepsilon > 0$ and $k \in \mathbb{N}$, with $k > N/2 + 1$. Consider a solution $(U_\varepsilon, \Phi_\varepsilon)$ to (H_ε), with $(U_\varepsilon, \Phi_\varepsilon) \in \mathcal{C}^0([0,T], \mathcal{NV}^{k+3}_{\sin}(\mathbb{R}^N))$ for some $T > 0$. Assume that*

$$\inf_{\mathbb{R}^N \times [0,T]} (1 - \varepsilon^2 U_\varepsilon^2) \geq \frac{1}{2}.$$

Then there exists $C > 0$, depending only on k and N, such that

$$\begin{aligned}\big[E_\varepsilon^\ell\big]'(t) \leq C \max\big(1, \sigma^{3/2}\big)\big(1 + \varepsilon^4\big) \Big(&\|\sin(\Phi_\varepsilon(\cdot,t))\|_{L^\infty}^2 + \|U_\varepsilon(\cdot,t)\|_{L^\infty}^2 + \|\nabla\Phi_\varepsilon(\cdot,t)\|_{L^\infty}^2 \\ &+ \|\nabla U_\varepsilon(\cdot,t)\|_{L^\infty}^2 + \|d^2\Phi_\varepsilon(\cdot,t)\|_{L^\infty}^2 + \varepsilon^2\|d^2 U_\varepsilon(\cdot,t)\|_{L^\infty}^2 \\ &+ \varepsilon\,\|\nabla\Phi_\varepsilon(\cdot,t)\|_{L^\infty}\big(\|\nabla\Phi_\varepsilon(\cdot,t)\|_{L^\infty}^2 + \|\nabla U_\varepsilon(\cdot,t)\|_{L^\infty}^2\big)\Big)\,\Sigma_\varepsilon^{k+1}(t),\end{aligned} \tag{2.11}$$

for any $t \in [0, T]$ and any $2 \leq \ell \leq k+1$. Here, we have set $\Sigma_\varepsilon^{k+1} := \sum_{j=1}^{k+1} E_\varepsilon^j$.

Thanks to the condition $k > N/2 + 1$ and the Sobolev embedding, we get from (2.11) a differential inequality for $y(t) := \Sigma_\varepsilon^k$, of the type

$$y'(t) \leq A y^2(t), \tag{2.12}$$

at least on the interval where y is well-defined and $y(t) \leq 2y(0)$. Here A is a constant depending on $y(0)$. Integrating (2.12), we conclude that

$$y(t) \leq \frac{y(0)}{1 - Ay(0)t} \leq 2y(0),$$

provided that $t \leq 1/(2Ay(0))$. Using this argument, we deduce from Proposition 2.5, that maximal time $T_{\max}$ is at least of order $1/(\|U_\varepsilon^0\|_{H^k} + \varepsilon\|\nabla U_\varepsilon^0\|_{H^k} + \|\nabla\Phi_\varepsilon^0\|_{H^k} + \|\sin(\Phi_\varepsilon^0)\|_{H^k})^2$, when the initial conditions $(U_\varepsilon^0, \Phi_\varepsilon^0)$ satisfy the inequality in (2.2). In particular, the dependence of $T_{\max}$ on the small parameter ε only results from the possible dependence of the pair $(U_\varepsilon^0, \Phi_\varepsilon^0)$ on ε. Choosing suitably these initial conditions, we can assume without loss of generality, that $T_{\max}$ is uniformly bounded from below when ε tends to 0, so that analyzing this limit makes sense. More precisely, we deduce from Proposition 2.5 the following results.

Corollary 2.6. *Let $\varepsilon > 0$ and $k \in \mathbb{N}$, with $k > N/2 + 1$. There exists $C > 0$, depending only on σ, k and N, such that if the initial data $(U_\varepsilon^0, \Phi_\varepsilon^0) \in \mathcal{NV}_{\sin}^{k+2}(\mathbb{R}^N)$ satisfies*

$$C\varepsilon\Big(\big\|U_\varepsilon^0\big\|_{H^k} + \varepsilon\big\|\nabla U_\varepsilon^0\big\|_{H^k} + \big\|\nabla\Phi_\varepsilon^0\big\|_{H^k} + \big\|\sin(\Phi_\varepsilon^0)\big\|_{H^k}\Big) \leq 1,$$

then there exists a positive time

$$T_\varepsilon \geq \frac{1}{C\big(\|U_\varepsilon^0\|_{H^k} + \varepsilon\|\nabla U_\varepsilon^0\|_{H^k} + \|\nabla\Phi_\varepsilon^0\|_{H^k} + \|\sin(\Phi_\varepsilon^0)\|_{H^k}\big)^2},$$

such that the unique solution $(U_\varepsilon, \Phi_\varepsilon)$ to (H_ε) with initial condition $(U_\varepsilon^0, \Phi_\varepsilon^0)$ satisfies the uniform bound

$$\varepsilon\big\|U_\varepsilon(\cdot,t)\big\|_{L^\infty} \leq \frac{1}{\sqrt{2}},$$

as well as the energy estimate

$$\begin{aligned}\big\|U_\varepsilon(\cdot,t)\big\|_{H^k} + \varepsilon\big\|\nabla U_\varepsilon(\cdot,t)\big\|_{H^k} + \big\|\nabla\Phi_\varepsilon(\cdot,t)\big\|_{H^k} + \big\|\sin(\Phi_\varepsilon(\cdot,t))\big\|_{H^k}& \\ \leq C\Big(\big\|U_\varepsilon^0\big\|_{H^k} + \varepsilon\big\|\nabla U_\varepsilon^0\big\|_{H^k} + \big\|\nabla\Phi_\varepsilon^0\big\|_{H^k} + \big\|\sin(\Phi_\varepsilon^0)\big\|_{H^k}\Big)&,\end{aligned} \tag{2.13}$$

for any $0 \leq t \leq T_\varepsilon$.

Remark 2.7. In the one-dimensional case, the conservation of the energy provides a much direct control on the quantity $\varepsilon\|U_\varepsilon\|_{L^\infty}$. This claim follows from the inequality

$$\varepsilon^2\|U_\varepsilon\|_{L^\infty}^2 \leq 2\varepsilon^2 \int_{\mathbb{R}} |U_\varepsilon'(x)|\,|U_\varepsilon(x)|\,dx \leq \varepsilon \int_{\mathbb{R}} \big(\varepsilon^2 U_\varepsilon'(x)^2 + U_\varepsilon(x)^2\big)\,dx.$$

When $\varepsilon\|U_\varepsilon^0\|_{L^\infty} < 1$, and the quantity $\varepsilon E_\varepsilon(0)$ is small enough, combining this inequality with the conservation of the energy E_ε and performing a continuity argument give a uniform control on the function $\varepsilon U_\varepsilon$ for any possible time.

As a further consequence of Proposition 2.5, Corollary 2.6 also provides the Sobolev control in (2.13) on the solution $(U_\varepsilon, \Phi_\varepsilon)$, which is uniform with respect to ε. This estimate is crucial in the proof of Theorem 2.1. As a matter of fact, the key ingredient in this proof is the consistency of (H_ε) with the Sine–Gordon system in the limit $\varepsilon \to 0$. Indeed, we can rewrite (H_ε) as

$$\begin{cases} \partial_t U_\varepsilon = \Delta\Phi_\varepsilon - \frac{\sigma}{2}\sin(2\Phi_\varepsilon) + \varepsilon^2 R_\varepsilon^U, \\ \partial_t \Phi_\varepsilon = U_\varepsilon + \varepsilon^2 R_\varepsilon^\Phi, \end{cases} \tag{2.14}$$

where we have set

$$R_\varepsilon^U := -\operatorname{div}\big(U_\varepsilon^2\,\nabla\Phi_\varepsilon\big) + \sigma U_\varepsilon^2\,\sin(\Phi_\varepsilon)\,\cos(\Phi_\varepsilon),$$

and

$$R_\varepsilon^\Phi := -\sigma U_\varepsilon\,\sin^2(\Phi_\varepsilon) - \operatorname{div}\Big(\frac{\nabla U_\varepsilon}{1-\varepsilon^2 U_\varepsilon^2}\Big) + \varepsilon^2 U_\varepsilon\,\frac{|\nabla U_\varepsilon|^2}{(1-\varepsilon^2U_\varepsilon^2)^2} - U_\varepsilon\,|\nabla\Phi_\varepsilon|^2.$$

In view of the Sobolev control in (2.13), the remainder terms R_ε^U and R_ε^Φ are bounded uniformly with respect to ε in Sobolev spaces, with a loss of three derivatives. Due to this observation, the differences $u_\varepsilon := U_\varepsilon - U$ and $\varphi_\varepsilon := \Phi_\varepsilon - \Phi$ between a solution $(U_\varepsilon, \Phi_\varepsilon)$ to (H_ε) and a solution (U, Φ) to (SGS) are expected to be of order ε^2, if the corresponding initial conditions are close enough.

The proof of this claim would be immediate if the system (2.14) would not contain the nonlinear term $\sin(2\Phi_\varepsilon)$. Due to this extra term, we have to apply a Gronwall argument in order to control the differences u_ε and φ_ε. Rolling out this argument requires an additional Sobolev control on the solution (U, Φ) to (SGS).

In this direction, we use the consistency of the systems (2.14) and (SGS) to mimic the proof of Corollary 2.6 for a solution (U, Φ) to (SGS). Indeed, when $\varepsilon = 0$, the quantities E_ε^k in (2.10) reduce to

$$E_{\mathrm{SG}}^k(U,\Phi) := \frac{1}{2}\sum_{|\alpha|=k-1}\int_{\mathbb{R}^N}\Big(|\partial_x^\alpha U|^2 + |\partial_x^\alpha\nabla\Phi|^2 + \sigma|\partial_x^\alpha\sin(\Phi)|^2\Big).$$

When (U, Φ) is a smooth enough solution to (SGS), we can perform energy estimates on these quantities in order to obtain

Lemma 2.8. *Let $k \in \mathbb{N}$, with $k > N/2 + 1$. There exists $A > 0$, depending only on σ, k and N, such that, given any initial data $(U^0, \Phi^0) \in H^{k-1}(\mathbb{R}^N) \times H^k_{\sin}(\mathbb{R}^N)$, there exists a positive time*

$$T_* \geq \frac{1}{A\big(\|U^0\|_{H^{k-1}} + \|\nabla\Phi^0\|_{H^{k-1}} + \|\sin(\Phi^0)\|_{H^{k-1}}\big)^2},$$

such that the unique solution (U, Φ) to (SGS) *with initial condition (U^0, Φ^0) satisfies the energy estimate*

$$\begin{aligned}\|U(\cdot,t)\|_{H^{k-1}} + \|\nabla\Phi(\cdot,t)\|_{H^{k-1}} + &\|\sin(\Phi(\cdot,t))\|_{H^{k-1}} \\ &\leq A\Big(\|U^0\|_{H^{k-1}} + \|\nabla\Phi^0\|_{H^{k-1}} + \|\sin(\Phi^0)\|_{H^{k-1}}\Big),\end{aligned}$$

for any $0 \leq t \leq T_*$.

In view of (SGS) and (2.14), the differences $v_\varepsilon = U_\varepsilon - U$ and $\varphi_\varepsilon = \Phi_\varepsilon - \Phi$ satisfy

$$\begin{cases}\partial_t v_\varepsilon = \Delta\varphi_\varepsilon - \sigma\sin(\varphi_\varepsilon)\,\cos(\Phi_\varepsilon + \Phi) + \varepsilon^2 R_\varepsilon^U, \\ \partial_t\varphi_\varepsilon = v_\varepsilon + \varepsilon^2 R_\varepsilon^\Phi.\end{cases}$$

With Corollary 2.6 and Lemma 2.8 at hand, we can control these differences by performing similar energy estimates on the functionals

$$\sum_{|\alpha|=k-1}\int_{\mathbb{R}^N}\big(|\partial_x^\alpha v_\varepsilon|^2 + |\partial_x^\alpha\nabla\varphi_\varepsilon|^2 + \sigma|\partial_x^\alpha\sin(\varphi_\varepsilon)|^2\big).$$

This is enough to obtain

Proposition 2.9. *Let $k \in \mathbb{N}$, with $k > N/2 + 1$. Given an initial condition $(U_\varepsilon^0, \Phi_\varepsilon^0) \in \mathcal{NV}^{k+2}_{\sin}(\mathbb{R}^N)$, assume that the unique corresponding solution $(U_\varepsilon, \Phi_\varepsilon)$ to* (H_ε) *is well-defined on a time interval $[0, T]$ for a positive number T, and that it satisfies the uniform bound*

$$\varepsilon\big\|U_\varepsilon(\cdot,t)\big\|_{L^\infty} \leq \frac{1}{\sqrt{2}},$$

for any $t \in [0, T]$. Consider similarly an initial condition $(U^0, \Phi^0) \in L^2(\mathbb{R}^N) \times H^1_{\sin}(\mathbb{R}^N)$, and denote by $(U, \Phi) \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}^N) \times H^1_{\sin}(\mathbb{R}^N))$ the unique corresponding solution to (SGS)*. Set $u_\varepsilon := U_\varepsilon - U$, $\varphi_\varepsilon := \Phi_\varepsilon - \Phi$, and*

$$\mathcal{K}_\varepsilon(T) := \max_{t\in[0,T]}\Big(\|U_\varepsilon(\cdot,t)\|_{H^k} + \varepsilon\|\nabla U_\varepsilon(\cdot,t)\|_{H^k} + \|\nabla\Phi_\varepsilon(\cdot,t)\|_{H^k} + \|\sin(\Phi_\varepsilon(\cdot,t))\|_{H^k}\Big).$$

(i) Assume that $\Phi_\varepsilon^0 - \Phi^0 \in L^2(\mathbb{R}^N)$. Then, there exists a positive number $C_1 > 0$, depending only on σ and N, such that

$$\|\varphi_\varepsilon(\cdot,t)\|_{L^2} \leq C_1\Big(\|\varphi_\varepsilon^0\|_{L^2} + \|v_\varepsilon^0\|_{L^2} + \varepsilon^2\,\mathcal{K}_\varepsilon(T)\,\big(1 + \varepsilon^2\mathcal{K}_\varepsilon(T)^2 + \mathcal{K}_\varepsilon(T)^3\big)\Big)\,e^{C_1 t},$$

for any $t \in [0, T]$.

(ii) Assume that $N \geq 2$, or that $N = 1$ and $k > N/2 + 2$. Then, there exists $C_2 > 0$, depending only on σ and N, such that

$$\|u_\varepsilon(\cdot,t)\|_{L^2} + \|\nabla\varphi_\varepsilon(\cdot,t)\|_{L^2} + \|\sin(\varphi_\varepsilon(\cdot,t))\|_{L^2} \leq C_2\Big(\|u_\varepsilon^0\|_{L^2} + \|\nabla\varphi_\varepsilon^0\|_{L^2} + \|\sin(\varphi_\varepsilon^0)\|_{L^2} + \varepsilon^2\,\mathcal{K}_\varepsilon(T)\,\big(1 + \varepsilon^2\mathcal{K}_\varepsilon(T)^2 + \mathcal{K}_\varepsilon(T)^3\big)\Big)\,e^{C_2 t},$$

for any $t \in [0,T]$.

(iii) Assume that $k > N/2 + 3$ and that the pair (U,Φ) belongs to $\mathcal{C}^0([0,T], H^k(\mathbb{R}^N) \times H^{k+1}_{\sin}(\mathbb{R}^N))$. Set

$$\kappa_\varepsilon(T) := \mathcal{K}_\varepsilon(T) + \max_{t\in[0,T]}\Big(\|U(\cdot,t)\|_{H^k} + \|\nabla\Phi(\cdot,t)\|_{H^k} + \|\sin(\Phi(\cdot,t))\|_{H^k}\Big).$$

Then, there exists a positive number C_k, depending only on σ, k and N, such that

$$\begin{aligned}&\|u_\varepsilon(\cdot,t)\|_{H^{k-3}} + \|\nabla\varphi_\varepsilon(\cdot,t)\|_{H^{k-3}} + \|\sin(\varphi_\varepsilon(\cdot,t))\|_{H^{k-3}}\\ &\quad\leq C_k\Big(\|u_\varepsilon^0\|_{H^{k-3}} + \|\nabla\varphi_\varepsilon^0\|_{H^{k-3}} + \|\sin(\varphi_\varepsilon^0)\|_{H^{k-3}}\\ &\qquad + \varepsilon^2\,\kappa_\varepsilon(T)\,\big(1 + \varepsilon^2\kappa_\varepsilon(T)^2 + (1+\varepsilon^2)\kappa_\varepsilon(T)^3\big)\Big)\,e^{C_k(1+\kappa_\varepsilon(T)^2)t},\end{aligned}$$

for any $t \in [0,T]$.

Finally, going to the proof of Theorem 2.1, in view of Corollaries 1.7 and 2.6, there exists $C > 0$, depending only on σ, k and N, for which, given any initial condition $(U_\varepsilon^0, \Phi_\varepsilon^0) \in \mathcal{NV}^{k+2}_{\sin}(\mathbb{R}^N)$ such that (2.2) holds, there exists $T_\varepsilon > 0$ satisfying (2.3) such that the unique solution $(U_\varepsilon, \Phi_\varepsilon)$ to (H$_\varepsilon$) with initial conditions $(U_\varepsilon^0, \Phi_\varepsilon^0)$ lies in $\mathcal{C}^0([0,T_\varepsilon], \mathcal{NV}^{k+1}_{\sin}(\mathbb{R}^N))$. Moreover, the quantity $\mathcal{K}_\varepsilon(T_\varepsilon)$ in Proposition 2.9 is bounded by

$$\mathcal{K}_\varepsilon(T_\varepsilon) \leq C\mathcal{K}_\varepsilon(0).$$

Enlarging if necessary the value of C, we then deduce statements (ii) and (iii) in Theorem 2.1 from statements (i) and (ii) in Proposition 2.9.

Similarly, given a pair $(U^0, \Phi^0) \in H^k(\mathbb{R}^N) \times H^{k+1}_{\sin}(\mathbb{R}^N))$, we infer from Theorem 2.4 and Lemma 2.8 the existence of a number T_ε^* such that (2.6) holds, and the unique solution (U,Φ) to (SGS) with initial conditions (U^0, Φ^0) is in $\mathcal{C}^0([0,T_\varepsilon^*], H^k(\mathbb{R}^N) \times H^{k+1}_{\sin}(\mathbb{R}^N))$. Statement (iv) in Theorem 2.1 then follows from statement (iii) in Proposition 2.9.

2.2 The cubic NLS regime

We now focus on the cubic Schrödinger equation, which is obtained in a regime of strong easy-axis anisotropy of equation (2.1). For this purpose, we consider a uniaxial material in the direction corresponding to the vector $e_2 = (0,1,0)$ and we fix the anisotropy parameters as

$$\lambda_1 = \lambda_3 = \frac{1}{\varepsilon}.$$

For this choice, the complex map $\check{m} = m_1 + im_3$ associated with a solution $\boldsymbol{m}$ of (2.1) satisfies [1]

$$\begin{cases} i\partial_t \check{m} + m_2 \Delta \check{m} - \check{m}\Delta m_2 - \frac{1}{\varepsilon} m_2 \check{m} = 0, \\ \partial_t m_2 - \langle i\check{m}, \Delta \check{m}\rangle_{\mathbb{C}} = 0. \end{cases}$$

Let us introduce the complex-valued function Ψ_ε given by

$$\Psi_\varepsilon(x,t) = \varepsilon^{-1/2} \check{m}(x,t) e^{it/\varepsilon}. \tag{2.15}$$

This function is of order 1 in the regime where the map $\check{m}$ is of order $\varepsilon^{\frac{1}{2}}$. When ε is small enough, the function m_2 does not vanish in this regime, since the solution $\boldsymbol{m}$ is valued into the sphere $\mathbb{S}^2$. Assuming that m_2 is everywhere positive, it is given by the formula

$$m_2 = \big(1 - \varepsilon|\Psi_\varepsilon|^2|\big)^{\frac{1}{2}},$$

and the function Ψ_ε is a solution to the nonlinear Schrödinger equation

$$i\partial_t \Psi_\varepsilon + \big(1 - \varepsilon|\Psi_\varepsilon|^2\big)^{1/2} \Delta \Psi_\varepsilon + \frac{|\Psi_\varepsilon|^2}{1 + (1 - \varepsilon|\Psi_\varepsilon|^2)^{1/2}} \Psi_\varepsilon + \varepsilon \operatorname{div}\Big(\frac{\langle \Psi_\varepsilon, \nabla \Psi_\varepsilon\rangle_{\mathbb{C}}}{(1 - \varepsilon|\Psi_\varepsilon|^2)^{1/2}}\Big)\Psi_\varepsilon = 0. \tag{NLS$_\varepsilon$}$$

As $\varepsilon \to 0$, the formal limit is therefore the focusing cubic Schrödinger equation

$$i\partial_t \Psi + \Delta \Psi + \frac{1}{2}|\Psi|^2 \Psi = 0. \tag{CS}$$

The main goal of this section is to justify rigorously this cubic Schrödinger regime of the LL equation.

We first recall a local well-posedness result for the Cauchy problem for the cubic Schrödinger equation, that can be obtained by using a fixed-point argument. We refer to [Caz03] for an extended review on this subject.

Theorem 2.10 ([Caz03])**.** *Let $k \in \mathbb{N}$, with $k > N/2$. Given any function $\Psi^0 \in H^k(\mathbb{R}^N)$, there exist $T_{\max} > 0$ and a unique solution $\Psi \in \mathcal{C}^0([0, T_{\max}), H^k(\mathbb{R}^N))$ to the cubic Schrödinger equation CS with initial data Ψ^0, which satisfies the following properties.*

(i) If the maximal time of existence $T_{\max}$ is finite, then

$$\lim_{t \to T_{\max}} \|\Psi(\cdot,t)\|_{H^k} = \infty, \quad \textit{and} \quad \limsup_{t \to T_{\max}} \|\Psi(\cdot,t)\|_{L^\infty} = \infty.$$

(ii) The flow map $\Psi^0 \mapsto \Psi$ is well-defined and Lipschitz continuous from $H^k(\mathbb{R}^N)$ to $\mathcal{C}^0([0,T], H^k(\mathbb{R}^N))$, for any $0 < T < T_{\max}$.

[1] Here as in the sequel, the notation $\langle z_1, z_2\rangle_{\mathbb{C}}$ stands for the canonical real scalar product of the two complex numbers z_1 and z_2, which is given by

$$\langle z_1, z_2\rangle_{\mathbb{C}} = \operatorname{Re}(z_1)\operatorname{Re}(z_2) + \operatorname{Im}(z_1)\operatorname{Im}(z_2) = \operatorname{Re}(z_1 \bar{z}_2).$$

(iii) When $\Psi^0 \in H^\ell(\mathbb{R}^N)$, with $\ell > k$, the solution Ψ lies in $\mathcal{C}^0([0,T], H^\ell(\mathbb{R}^N))$, for any $0 < T < T_{\max}$.

(iv) The L^2-mass M_2 and the cubic Schrödinger energy $E_{\rm CS}$ given by

$$M_2(\Psi) = \int_{\mathbb{R}^N} |\Psi|^2, \quad \textit{and} \quad E_{\rm CS}(\Psi) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\Psi|^2 - \frac{1}{4}\int_{\mathbb{R}^N} |\Psi|^4,$$

are conserved along the flow.

Going on with our rigorous derivation of the cubic Schrödinger regime, we now express the local well-posedness result in Theorem 1.2 in terms of the nonlinear Schrödinger equation (NLS_ε) satisfied by the rescaled function Ψ_ε.

Corollary 2.11 ([dLG21]). *Let $\varepsilon > 0$, and $k \in \mathbb{N}$, with $k > N/2 + 1$. Consider a function $\Psi_\varepsilon^0 \in H^k(\mathbb{R}^N)$ such that*

$$\varepsilon^{1/2} \left\|\Psi_\varepsilon^0\right\|_{L^\infty} < 1. \tag{2.16}$$

Then, there exist $T_\varepsilon > 0$ and a unique solution $\Psi_\varepsilon : \mathbb{R}^N \times [0, T_\varepsilon) \to \mathbb{C}$ to (NLS_ε) with initial data Ψ_ε^0, which satisfies the following properties.

(i) The solution Ψ_ε is in the space $L^\infty([0,T], H^k(\mathbb{R}^N))$, while its time derivative $\partial_t\Psi_\varepsilon$ is in $L^\infty([0,T], H^{k-2}(\mathbb{R}^N))$, for any $0 < T < T_\varepsilon$.

(ii) If the maximal time of existence T_ε is finite, then

$$\int_0^{T_\varepsilon} \left\|\nabla\Psi_\varepsilon(\cdot,t)\right\|_{L^\infty}^2 dt = \infty, \quad \textit{or} \quad \varepsilon^{1/2} \lim_{t\to T_\varepsilon} \left\|\Psi_\varepsilon(\cdot,t)\right\|_{L^\infty} = 1.$$

(iii) The flow map $\Psi_\varepsilon^0 \mapsto \Psi_\varepsilon$ is locally well-defined and Lipschitz continuous from $H^k(\mathbb{R}^N)$ to $\mathcal{C}^0([0,T], H^{k-1}(\mathbb{R}^N))$ for any $0 < T < T_\varepsilon$.

(iv) When $\Psi_\varepsilon^0 \in H^\ell(\mathbb{R}^N)$, with $\ell > k$, the solution Ψ_ε lies in $L^\infty([0,T], H^\ell(\mathbb{R}^N))$, with $\partial_t\Psi_\varepsilon \in L^\infty([0,T], H^{\ell-2}(\mathbb{R}^N))$ for any $0 < T < T_\varepsilon$.

(v) The nonlinear Schrödinger energy $\mathfrak{E}_\varepsilon$ given by

$$\mathfrak{E}_\varepsilon(\Psi_\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\Psi_\varepsilon|^2 + \varepsilon|\nabla\Psi_\varepsilon|^2 + \frac{\varepsilon^2\langle\Psi_\varepsilon, \nabla\Psi_\varepsilon\rangle_{\mathbb{C}}^2}{1 - \varepsilon|\Psi_\varepsilon|^2} \right),$$

is conserved along the flow.

(vi) Set

$$\boldsymbol{m}^0 = \Big(\varepsilon^{\frac{1}{2}}\ \mathrm{Re}\,\big(\Psi_\varepsilon^0\big), \big(1 - \varepsilon|\Psi_\varepsilon^0|^2\big)^{\frac{1}{2}}, \varepsilon^{\frac{1}{2}}\ \mathrm{Im}\,\big(\Psi_\varepsilon^0\big)\Big).$$

The function $\boldsymbol{m} : \mathbb{R}^N \times [0, T_\varepsilon] \to \mathbb{S}^2$ given by

$$\boldsymbol{m}(x,t) = \Big(\varepsilon^{\frac{1}{2}}\ \mathrm{Re}\,\big(e^{-\frac{it}{\varepsilon}}\Psi_\varepsilon(x,t)\big), \big(1 - \varepsilon|\Psi_\varepsilon(x,t)|^2\big)^{\frac{1}{2}}, \varepsilon^{\frac{1}{2}}\ \mathrm{Im}\,\big(e^{-\frac{it}{\varepsilon}}\Psi_\varepsilon(x,t)\big)\Big),$$

for any $(x,t) \in \mathbb{R}^N \times [0, T_\varepsilon]$, is the unique solution to (2.1) *with initial data $\boldsymbol{m}^0$ of Theorem 1.2, with $\lambda_1 = \lambda_3 = 1/\varepsilon$.*

With Corollary 2.11 at hand, we are now in position to state our main result concerning the rigorous derivation of the cubic Schrödinger regime of the LL equation.

Theorem 2.12 ([dLG21])**.** *Let $0 < \varepsilon < 1$, and $k \in \mathbb{N}$, with $k > N/2+2$. Consider two initial conditions $\Psi^0 \in H^k(\mathbb{R}^N)$ and $\Psi^0_\varepsilon \in H^{k+3}(\mathbb{R}^N)$, and set*

$$\mathcal{K}_\varepsilon := \big\|\Psi^0\big\|_{H^k} + \big\|\Psi^0_\varepsilon\big\|_{H^k} + \varepsilon^{\frac{1}{2}}\big\|\nabla\Psi^0_\varepsilon\big\|_{\dot{H}^k} + \varepsilon\big\|\Delta\Psi^0_\varepsilon\big\|_{\dot{H}^k}.$$

There exists $A > 0$, depending only on k, such that, if the initial data Ψ^0 and Ψ^0_ε satisfy the condition

$$A\,\varepsilon^{\frac{1}{2}}\,\mathcal{K}_\varepsilon \leq 1, \tag{2.17}$$

then the following statements hold.

(i) There exists a time

$$T_\varepsilon \geq \frac{1}{A\mathcal{K}_\varepsilon^2},$$

such that both the unique solution Ψ_ε to (NLS_ε) *with initial data Ψ^0_ε, and the unique solution Ψ to* (CS) *with initial data Ψ^0 are well-defined on the time interval $[0, T_\varepsilon]$.*

(ii) We have the error estimate

$$\big\|\Psi_\varepsilon(\cdot,t) - \Psi(\cdot,t)\big\|_{H^{k-2}} \leq \Big(\big\|\Psi^0_\varepsilon - \Psi^0\big\|_{H^{k-2}} + A\varepsilon\mathcal{K}_\varepsilon\big(1+\mathcal{K}_\varepsilon^3\big)\Big)\, e^{A\mathcal{K}_\varepsilon^2 t}, \tag{2.18}$$

for any $0 \leq t \leq T_\varepsilon$.

In this manner, Theorem 2.12 establish rigorously the convergence of the LL equation towards the cubic Schrödinger equation in any dimension. The assumption $k > N/2 + 2$ in Theorem 2.12 comes from our choice to quantify this convergence. Our estimates are taylored in order to obtain the ε factor in the right-hand side of the error estimate (2.18), since we expect this order of convergence to be sharp. More precisely, using the solitons for the LL equation, we can compute explicitly solitons for (NLS_ε) and then prove that their difference with respect to the corresponding bright solitons $\Psi_{c,\omega}$ of the cubic Schrödinger equation is of exact order ε, as the error factor in (2.18) (see [dLG21]).

It is certainly possible to show only convergence under weaker assumptions by using compactness arguments as for the derivation of similar asymptotic regimes (see e.g. [SZ06, CR10, GR19] concerning Schrödinger-like equations).

Observe that smooth solutions for both the LL and the cubic Schrödinger equations are known to exist when the integer k satisfies the condition $k > N/2 + 1$. The additional assumption $k > N/2+2$ in Theorem 2.12 is related to the fact that our proof of (2.18) requires a uniform control of the difference $\Psi_\varepsilon - \Psi$, which follows from the Sobolev embedding theorem of $H^{k-2}(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$.

Similarly, the fact that Ψ^0_ε is taken in $H^{k+3}(\mathbb{R}^N)$ instead of $H^{k+2}(\mathbb{R}^N)$, which is enough to define the quantity $\mathcal{K}_\varepsilon$, is related to the loss of one derivative for establishing the flow continuity in statement *(iii)* of Corollary 2.11.

Finally, the loss of two derivatives in the error estimate (2.18) can be partially recovered by combining standard interpolation theory with the estimates in Proposition 2.16 and Lemma 2.17. Under the assumptions of Theorem 2.12, the solutions Ψ_ε converge towards the solution Ψ in $\mathcal{C}^0([0,T_\varepsilon], H^s(\mathbb{R}^N))$ for any $0 \leq s < k$, when Ψ_ε^0 tends to Ψ^0 in $H^{k+2}(\mathbb{R}^N)$ as $\varepsilon \to 0$, but the error term is not necessarily of order ε due to the interpolation process.

Note here that condition (2.17) is not really restrictive in order to analyze such a convergence. At least when Ψ_ε^0 tends to Ψ^0 in $H^{k+2}(\mathbb{R}^N)$ as $\varepsilon \to 0$, the quantity $\mathcal{K}_\varepsilon$ tends to twice the norm $\|\Psi^0\|_{H^k}$ in the limit $\varepsilon \to 0$, so that condition (2.17) is always fulfilled. Moreover, the error estimate (2.18) is available on a time interval of order $1/\|\Psi^0\|_{H^k}^2$, which is similar to the minimal time of existence of the smooth solutions to the cubic Schrödinger equation (see Lemma 2.17 below).

2.3 Sketch of the proof of Theorem 2.12

The proof relies on the consistency between the Schrödinger equations (NLS_ε) and (CS) in the limit $\varepsilon \to 0$. Indeed, we can recast (NLS_ε) as

$$i\partial_t \Psi_\varepsilon + \Delta\Psi_\varepsilon + \frac{1}{2}|\Psi_\varepsilon|^2\Psi_\varepsilon = \varepsilon\mathcal{R}_\varepsilon, \tag{2.19}$$

where the remainder term $\mathcal{R}_\varepsilon$ is given by

$$\mathcal{R}_\varepsilon := \frac{|\Psi_\varepsilon|^2}{1 + (1 - \varepsilon|\Psi_\varepsilon|^2)^{\frac{1}{2}}}\Delta\Psi_\varepsilon - \frac{|\Psi_\varepsilon|^4}{2(1 + (1 - \varepsilon|\Psi_\varepsilon|^2)^{\frac{1}{2}})^2}\Psi_\varepsilon - \operatorname{div}\Big(\frac{\langle\Psi_\varepsilon, \nabla\Psi_\varepsilon\rangle_\mathbb{C}}{(1 - \varepsilon|\Psi_\varepsilon|^2)^{\frac{1}{2}}}\Big)\Psi_\varepsilon. \tag{2.20}$$

In order to establish the convergence towards the cubic Schrödinger equation, our main goal is to control the remainder term $\mathcal{R}_\varepsilon$ on a time interval $[0, T_\varepsilon]$ as long as possible. In particular, we have to show that the maximal time T_ε for this control does not vanish in the limit $\varepsilon \to 0$.

The strategy for reaching this goal is reminiscent from a series of papers concerning the rigorous derivation of long-wave regimes for various Schrödinger-like equations (see [SZ06, BDS09, BGSS09, CR10, BGSS10, BDG$^+$10, Chi14, GR19, dLG18] and the references therein). The main argument is to perform suitable energy estimates on the solutions Ψ_ε to (NLS_ε). These estimates provide Sobolev bounds for the remainder term $\mathcal{R}_\varepsilon$, which are used to control the differences $u_\varepsilon := \Psi_\varepsilon - \Psi$ with respect to the solutions Ψ to (CS). This further control is also derived from energy estimates.

Concerning the estimates of the solutions Ψ_ε, we rely on the equivalence with the solutions $\boldsymbol{m}$ to (2.1). However, the estimates given in Chapter 1 are not enough in this case. It is crucial to refine the estimate (1.6), which will be done by using that $\lambda_1 = \lambda_3$. More precisely, given a positive number T and a sufficiently smooth solution $\boldsymbol{m} : \mathbb{R}^N \times [0,T] \to \mathbb{S}^2$ to (2.1), we recall that the energy E_{LL}^k of order $k \geq 2$ is given by

$$\begin{aligned} E_{\text{LL}}^k(t) := \frac{1}{2}\Big(&\|\partial_t m(\cdot,t)\|_{\dot{H}^{k-2}}^2 + \|\Delta m(\cdot,t)\|_{\dot{H}^{k-2}}^2 + (\lambda_1 + \lambda_3)\big(\|\nabla m_1(\cdot,t)\|_{\dot{H}^{k-2}}^2 \\ &+ \|\nabla m_3(\cdot,t)\|_{\dot{H}^{k-2}}^2\big) + \lambda_1\lambda_3\big(\|m_1(\cdot,t)\|_{\dot{H}^{k-2}}^2 + \|m_3(\cdot,t)\|_{\dot{H}^{k-2}}^2\big)\Big), \end{aligned}$$

for any $t \in [0,T]$. Then, in the regime $\lambda_1 = \lambda_3 = 1/\varepsilon$, we can prove the following improvement of the computations made in Proposition 1.5.

Proposition 2.13. *Let $0 < \varepsilon < 1$, and $k \in \mathbb{N}$, with $k > N/2 + 1$. Assume that*

$$\lambda_1 = \lambda_3 = \frac{1}{\varepsilon},$$

and that $\boldsymbol{m}$ is a solution to (2.1) in $\mathcal{C}^0([0,T], \mathcal{E}^{k+4}(\mathbb{R}^N))$, with $\partial_t m \in \mathcal{C}^0([0,T], H^{k+2}(\mathbb{R}^N))$. Given any integer $2 \leq \ell \leq k+2$, the energies $E_{\rm LL}^\ell$ are of class $\mathcal{C}^1$ on $[0,T]$, and there exists $C_k > 0$, depending possibly on k, but not on ε, such that their derivatives satisfy

$$\big[E_{\rm LL}^\ell\big]'(t) \leq \frac{C_k}{\varepsilon}\Big(\|m_1(\cdot,t)\|_{L^\infty}^2 + \|m_3(\cdot,t)\|_{L^\infty}^2 + \|\nabla m(\cdot,t)\|_{L^\infty}^2\Big)\Big(E_{\rm LL}^\ell(t) + E_{\rm LL}^{\ell-1}(t)\Big), \quad (2.21)$$

for any $t \in [0,T]$. Here we have set $E_{\rm LL}^1(t) := E_{\rm LL}(m(\cdot,t))$, the LL energy.

As for the proof of Proposition 1.5, the estimates in Proposition 2.13 rely on the identity (1.5), that in the case $\lambda_1 = \lambda_3 = 1/\varepsilon$ can be simplified as

$$\partial_{tt} m + \Delta^2 m - \frac{2}{\varepsilon}\Big(\Delta m_1 e_1 + \Delta m_3 e_3\Big) + \frac{1}{\varepsilon^2}\Big(m_1 e_1 + m_3 e_3\Big) = F_\varepsilon(m), \quad (2.22)$$

where

$$\begin{aligned} F_\varepsilon(m) := \sum_{1\leq i,j\leq N} &\Big(\partial_i\big(2\langle\partial_i m, \partial_j m\rangle_{\mathbb{R}^3}\partial_j m - |\partial_j m|^2\partial_i m\big) - 2\partial_{ij}\big(\langle\partial_i m, \partial_j m\rangle_{\mathbb{R}^3} m\big)\Big) \\ &- \frac{1}{\varepsilon}\Big((m_1^2 + 3m_3^2)\Delta m_1 e_1 + (3m_1^2 + m_3^2)\Delta m_3 e_3 - 2m_1 m_3(\Delta m_1 e_3 + \Delta m_3 e_1) \\ &\quad + (m_1^2 + m_3^2)\Delta m_2 e_2 - |\nabla m|^2(m_1 e_1 + m_3 e_3) + \nabla\big(m_1^2 + m_3^2\big)\cdot\nabla m\Big) \\ &+ \frac{1}{\varepsilon^2}\Big((m_1^2 + m_3^2)(m_1 e_1 + m_3 e_3)\Big). \end{aligned}$$

Using (2.22), the proof of Proposition 2.13 follows as in Proposition 1.5. However, in contrast with the estimate (1.6), the multiplicative factor in the right-hand side of (2.21) now only depends on the uniform norms of the functions m_1, m_3 and $\nabla \boldsymbol{m}$. This property is key in order to use these estimates in the cubic Schrödinger regime.

The next step of the proof is indeed to express the quantities $E_{\rm LL}^k$ in terms of the functions Ψ_ε. Assume that these functions $\Psi_\varepsilon : \mathbb{R}^N \times [0,T] \to \mathbb{C}$ are smooth enough. In view of (2.15) and (NLS$_\varepsilon$), we propose the following high order energy

$$\begin{aligned} \mathfrak{E}_\varepsilon^k(t) := \frac{1}{2}\Big(&\big\|\Psi_\varepsilon(\cdot,t)\big\|_{\dot H^{k-2}}^2 + \big\|\varepsilon\partial_t\Psi_\varepsilon(\cdot,t) - i\Psi_\varepsilon(\cdot,t)\big\|_{\dot H^{k-2}}^2 + \varepsilon^2\big\|\Delta\Psi_\varepsilon(\cdot,t)\big\|_{\dot H^{k-2}}^2 \\ &+\varepsilon\Big(\big\|\partial_t(1 - \varepsilon|\Psi_\varepsilon(\cdot,t)|^2)^{\frac{1}{2}}\big\|_{\dot H^{k-2}}^2 + \big\|\Delta(1 - \varepsilon|\Psi_\varepsilon(\cdot,t)|^2)^{\frac{1}{2}}\big\|_{\dot H^{k-2}}^2 + 2\big\|\nabla\Psi_\varepsilon(\cdot,t)\big\|_{\dot H^{k-2}}^2\Big)\Big), \end{aligned}$$

for any $k \geq 2$ and any $t \in [0,T]$. Combining the local well-posedness result of Corollary 2.11 and the computations in Proposition 2.13, we obtain

Corollary 2.14. *Let $0 < \varepsilon < 1$, and $k \in \mathbb{N}$, with $k > N/2 + 1$. Consider $\Psi_\varepsilon^0 \in H^{k+5}(\mathbb{R}^N)$ satisfying condition (2.16), and let $\Psi_\varepsilon : \mathbb{R}^N \times [0, T_\varepsilon) \to \mathbb{C}$ be the corresponding solution to* (NLS_ε) *given by Corollary 2.11. Given any integer $2 \leq \ell \leq k+2$ and any number $0 \leq T < T_\varepsilon$, the energies $\mathfrak{E}_\varepsilon^\ell$ are of class $\mathcal{C}^1$ on $[0, T]$, and there exists a positive number C_k, depending possibly on k, but not on ε, such that their derivatives satisfy*

$$\big[\mathfrak{E}_\varepsilon^\ell\big]'(t) \leq C_k\bigg(\|\Psi_\varepsilon(\cdot,t)\|_{L^\infty}^2 + \|\nabla\Psi_\varepsilon(\cdot,t)\|_{L^\infty}^2 + \varepsilon\Big\|\frac{\langle\Psi_\varepsilon(\cdot,t), \nabla\Psi_\varepsilon(\cdot,t)\rangle_{\mathbb{C}}}{(1-\varepsilon|\Psi_\varepsilon(\cdot,t)|^2)^{\frac{1}{2}}}\Big\|_{L^\infty}^2\bigg)\big(\mathfrak{E}_\varepsilon^\ell(t) + \varepsilon^{\delta_{\ell,2}}\mathfrak{E}_\varepsilon^{\ell-1}(t)\big), \tag{2.23}$$

for any $t \in [0, T]$. Here we have set $\mathfrak{E}_\varepsilon^1(t) := \mathfrak{E}_\varepsilon(\Psi_\varepsilon(\cdot, t))$.

In order to gain a control on the solutions Ψ_ε to (NLS_ε) from inequality (2.23), we now have to characterize the Sobolev norms, which are controlled by the energies $\mathfrak{E}_\varepsilon^k$. In this direction, we show

Lemma 2.15. *Let $0 < \varepsilon < 1$, $T > 0$ and $k \in \mathbb{N}$, with $k > N/2 + 1$. Consider a solution $\Psi_\varepsilon \in \mathcal{C}^0([0, T], H^{k+4}(\mathbb{R}^N))$ to* (NLS_ε) *such that*

$$\sigma_T := \varepsilon^{\frac{1}{2}} \max_{t \in [0,T]} \big\|\Psi_\varepsilon(\cdot, t)\big\|_{L^\infty} < 1.$$

There exists $C > 0$, depending possibly on σ_T and k, but not on ε, such that

$$\begin{aligned}\frac{1}{2}\Big(\big\|\Psi_\varepsilon(\cdot,t)\big\|_{\dot{H}^{\ell-2}}^2 + \varepsilon\big\|\nabla\Psi_\varepsilon(\cdot,t)\big\|_{\dot{H}^{\ell-2}}^2 + \varepsilon^2\big\|\Delta\Psi_\varepsilon(\cdot,t)\big\|_{\dot{H}^{\ell-2}}^2\Big)& \\ \leq \mathfrak{E}_\varepsilon^\ell(t) \leq C\Big(\big\|\Psi_\varepsilon(\cdot,t)\big\|_{\dot{H}^{\ell-2}}^2 + \varepsilon\big\|\nabla\Psi_\varepsilon(\cdot,t)\big\|_{\dot{H}^{\ell-2}}^2 + \varepsilon^2\big\|\Delta\Psi_\varepsilon(\cdot,t)\big\|_{\dot{H}^{\ell-2}}^2\Big)&,\end{aligned}$$

for any $2 \leq \ell \leq k+2$ and any $t \in [0, T]$. Moreover,

$$\frac{1}{2}\Big(\big\|\Psi_\varepsilon(\cdot,t)\big\|_{L^2}^2 + \varepsilon\big\|\nabla\Psi_\varepsilon(\cdot,t)\big\|_{L^2}^2\Big) \leq \mathfrak{E}_\varepsilon^1(t) \leq C\Big(\big\|\Psi_\varepsilon(\cdot,t)\big\|_{L^2}^2 + \varepsilon\big\|\nabla\Psi_\varepsilon(\cdot,t)\big\|_{L^2}^2\Big),$$

for any $t \in [0, T]$.

With Corollary 2.14 and Lemma 2.15 at hand, we are now in position to provide the following control on the solutions Ψ_ε to (NLS_ε).

Proposition 2.16. *Let $0 < \varepsilon < 1$, $0 < \delta < 1$ and $k \in \mathbb{N}$, with $k > N/2 + 1$. There exists $C > 0$, depending possibly on δ and k, but not on ε, such that if an initial data $\Psi_\varepsilon^0 \in H^{k+3}(\mathbb{R}^N)$ satisfies*

$$C\varepsilon^{\frac{1}{2}}\Big(\big\|\Psi_\varepsilon^0\big\|_{H^k} + \varepsilon^{\frac{1}{2}}\big\|\nabla\Psi_\varepsilon^0\big\|_{\dot{H}^k} + \varepsilon\big\|\Delta\Psi_\varepsilon^0\big\|_{\dot{H}^k}\Big) \leq 1,$$

then there exists a positive time

$$T_\varepsilon \geq \frac{1}{C\big(\big\|\Psi_\varepsilon^0\big\|_{H^k} + \varepsilon^{\frac{1}{2}}\big\|\nabla\Psi_\varepsilon^0\big\|_{\dot{H}^k} + \varepsilon\big\|\Delta\Psi_\varepsilon^0\big\|_{\dot{H}^k}\big)^2},$$

such that the unique solution Ψ_ε to (NLS$_\varepsilon$) *with initial condition Ψ_ε^0 satisfies the uniform bound*

$$\varepsilon^{\frac{1}{2}} \big\| \Psi_\varepsilon(\cdot, t) \big\|_{L^\infty} \leq \delta,$$

as well as the energy estimate

$$\begin{aligned} \big\| \Psi_\varepsilon(\cdot, t) \big\|_{H^k} + \varepsilon^{\frac{1}{2}} \big\| \nabla \Psi_\varepsilon(\cdot, t) \big\|_{\dot{H}^k} + \varepsilon \big\| \Delta \Psi_\varepsilon(\cdot, t) \big\|_{\dot{H}^k} \\ \leq C \Big(\big\| \Psi_\varepsilon^0 \big\|_{H^k} + \varepsilon^{\frac{1}{2}} \big\| \nabla \Psi_\varepsilon^0 \big\|_{\dot{H}^k} + \varepsilon \big\| \Delta \Psi_\varepsilon^0 \big\|_{\dot{H}^k} \Big), \end{aligned} \tag{2.24}$$

for any $0 \leq t \leq T_\varepsilon$.

An important feature of Proposition 2.16 lies in the fact that the solutions Ψ_ε are controlled uniformly with respect to the small parameter ε up to a loss of three derivatives. This loss is usual in the context of asymptotic regimes for Schrödinger-like equations (see e.g [BGSS09, BGSS10] and the references therein). It is related to the property that the energies $E_{\rm LL}^k$ naturally scale according to the right-hand side of (2.24) in the limit $\varepsilon \to 0$. This property is the origin of a loss of two derivatives. The extra loss is due to the requirement to use the continuity of the (NLS$_\varepsilon$) flow with respect to the initial data in order to prove Proposition 2.16, and this continuity holds with a loss of one derivative in view of statement (*iii*) in Corollary 2.11.

We now turn to our ultimate goal, which is to estimate the error between a solution Ψ_ε to (NLS$_\varepsilon$) and a solution Ψ to (CS). Going back to (2.19), we check that their difference $u_\varepsilon := \Psi_\varepsilon - \Psi$ satisfies the equation

$$i \partial_t u_\varepsilon + \Delta u_\varepsilon + \frac{1}{2} \big(|u_\varepsilon + \Psi|^2 (u_\varepsilon + \Psi) - |\Psi|^2 \Psi \big) = \varepsilon \mathcal{R}_\varepsilon.$$

In view of (2.20), we can invoke Proposition 2.16 in order to bound the remainder term $\mathcal{R}_\varepsilon$ in suitable Sobolev norms. On the other hand, we also have to provide a Sobolev control of the solution Ψ to (CS) on a time interval as long as possible. In this direction, we can show the following classical result (see e.g. [Caz03]), by performing standard energy estimates on the H^k-norms of the solution Ψ.

Lemma 2.17. *Let $k \in \mathbb{N}$, with $k > N/2$, and $\Psi^0 \in H^k(\mathbb{R}^N)$. There exists a positive number C_k, depending possibly on k, such that there exists a positive time*

$$T_* \geq \frac{1}{C_k \| \Psi^0 \|_{H^k}^2},$$

for which the unique solution Ψ to (CS) *with initial condition Ψ^0 satisfies the energy estimate*

$$\big\| \Psi(\cdot, t) \big\|_{H^k} \leq C_k \big\| \Psi^0 \big\|_{H^k},$$

for any $0 \leq t \leq T_*$.

The final ingredient to complete the proof of Theorem 2.12 is to obtain energy estimates in order to control the difference u_ε, according to the following statement.

Proposition 2.18. *Let $0 < \varepsilon < 1$, $0 < \delta < 1$ and $k \in \mathbb{N}$, with $k > N/2 + 2$. Given an initial condition $\Psi_\varepsilon^0 \in H^{k+3}(\mathbb{R}^N)$, assume that the unique corresponding solution Ψ_ε to* (NLS$_\varepsilon$) *is well-defined on a time interval $[0, T]$ for some positive number T, and that it satisfies the uniform bound*

$$\varepsilon^{\frac{1}{2}} \big\| \Psi_\varepsilon(\cdot, t) \big\|_{L^\infty} \leq \delta,$$

for any $t \in [0, T]$. Assume similarly that the solution Ψ to (CS) *with initial data $\Psi^0 \in H^k(\mathbb{R}^N)$ is well-defined on $[0, T]$. Set $u_\varepsilon := \Psi_\varepsilon - \Psi$ and*

$$\mathcal{K}_\varepsilon(T) := \big\| \Psi \big\|_{\mathcal{C}^0([0,T],H^k)} + \big\| \Psi_\varepsilon \big\|_{\mathcal{C}^0([0,T],H^k)} + \varepsilon^{\frac{1}{2}} \big\| \nabla \Psi_\varepsilon \big\|_{\mathcal{C}^0([0,T],\dot{H}^k)} + \varepsilon \big\| \Delta \Psi_\varepsilon \big\|_{\mathcal{C}^0([0,T],\dot{H}^k)}.$$

Then there exists $C > 0$, depending possibly on δ and k, but not on ε, such that

$$\big\| u_\varepsilon(\cdot, t) \big\|_{H^{k-2}} \leq \Big(\big\| u_\varepsilon(\cdot, 0) \big\|_{H^{k-2}} + \varepsilon \mathcal{K}_\varepsilon(T) \big(1 + \mathcal{K}_\varepsilon(T)^3 \big) \Big) \, e^{C \mathcal{K}_\varepsilon(T)^2 t},$$

for any $t \in [0, T]$.

Chapter 3

Stability of sum of solitons

We consider in this chapter the one-dimensional easy-plane LL equation, that is (1.9) with $\lambda_3 > 0$. Thus, we can assume that $\lambda_3 = 1$ and we simply write

$$\partial_t \boldsymbol{m} + \boldsymbol{m} \times (\partial_{xx} \boldsymbol{m} - \lambda m_3 \boldsymbol{e}_3) = 0, \tag{3.1}$$

We focus on the analysis of localized solutions such as solitons. The main goal is to explain the results in [dLG15a], that provide the orbital stability of arbitrary perturbations of a (well-prepared) sum of solitons.

3.1 Sum of solitons and the hydrodynamical formulation

As seen in §1, there are exact soliton to equation (3.1) of the form $\boldsymbol{m}(x,t) = \boldsymbol{u}(x - ct)$. Moreover, these solutions are nonconstant with finite energy if $|c| < 1$, and there are given by

$$\boldsymbol{u}_c(x) = \big(c \operatorname{sech}\big(\sqrt{1-c^2}x\big), \tanh\big(\sqrt{1-c^2}x\big), \sqrt{1-c^2} \operatorname{sech}\big(\sqrt{1-c^2}x\big)\big),$$

up to the invariances of the equation, i.e. translations, rotations around the axis x_3 and orthogonal symmetries with respect to any line in the plane $x_3 = 0$. A nonconstant soliton with speed c may be written as

$$\boldsymbol{u}_{c,a,\theta,s}(x) = \big(\cos(\theta)[\boldsymbol{u}_c]_1 - s\sin(\theta)[\boldsymbol{u}_c]_2, \sin(\theta)[\boldsymbol{u}_c]_1 + s\cos(\theta)[\boldsymbol{u}_c]_2, s[\boldsymbol{u}_c]_3\big)(x-a),$$

with $a \in \mathbb{R}$, $\theta \in \mathbb{R}$ and $s \in \{\pm 1\}$.

In addition, using the integrability of the equation and by means of the inverse scattering method, for any $M \in \mathbb{N}^*$, it can be also computed explicit solutions to (3.1) that behave like a sum M decoupled solitons as $t \to \infty$. These solutions are often called M-solitons or simply multisolitons (see e.g. [BBI14, Section 10] for their explicit formula).

We can define properly the solitons in the hydrodynamical framework when $c \neq 0$, since the function $\check{u}_c = [\boldsymbol{u}_c]_1 + i[\boldsymbol{u}_c]_2$ does not vanish. More precisely, we recall that for a function

$\boldsymbol{u} : \mathbb{R} \to \mathbb{S}^2$ such that $|\boldsymbol{u}| \neq 0$, we set

$$\check{u} = (1 - u_3^2)^{1/2} \exp i\varphi,$$

and we define the hydrodynamical variables $v := u_3$ and $w := \partial_x \varphi$. Thus, equation (3.1) recast as

$$\begin{cases} \partial_t v = \partial_x\big((v^2-1)w\big), \\ \partial_t w = \partial_x\Big(\dfrac{\partial_{xx} v}{1-v^2} + v\dfrac{(\partial_x v)^2}{(1-v^2)^2} + v\big(w^2-1\big)\Big), \end{cases} \tag{3.2}$$

and the soliton $\boldsymbol{u}_c$ in the hydrodynamical variables $\mathfrak{v}_c := (v_c, w_c)$ is given by

$$v_c(x) = \sqrt{1-c^2}\,\mathrm{sech}\,\big(\sqrt{1-c^2}x\big), \text{ and } w_c(x) = \frac{c\,v_c(x)}{1-v_c(x)^2} = \frac{c\sqrt{1-c^2}\cosh\big(\sqrt{1-c^2}x\big)}{\sinh\big(\sqrt{1-c^2}x\big)^2 + c^2}. \tag{3.3}$$

In this framework, the only remaining invariances of solitons are translations, as well as the opposite map $(v, w) \mapsto (-v, -w)$. Any soliton with speed c may be then written as

$$\mathfrak{v}_{c,a,s}(x) := s\,\mathfrak{v}_c(x-a) := \big(s\,v_c(x-a), s\,w_c(x-a)\big),$$

with $a \in \mathbb{R}$ and $s \in \{\pm 1\}$.

Our goal in this chapter is to establish the orbital stability of a single soliton u_c along the LL flow. More generally, we will also consider the case of a sum of solitons. In the original framework, defining this sum is not so easy, since the sum of unit vectors in $\mathbb{R}^3$ does not necessarily remain in $\mathbb{S}^2$. In the hydrodynamical framework, this difficulty does not longer arise. We can define a sum of M solitons $S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ as

$$\boldsymbol{S}_{\mathfrak{c},\mathfrak{a},\mathfrak{s}} := (V_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}, W_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}) := \sum_{j=1}^{M} \mathfrak{v}_{c_j,a_j,s_j},$$

with $M \in \mathbb{N}^*$, $\mathfrak{c} = (c_1, \ldots, c_M)$, $\mathfrak{a} = (a_1, \ldots, a_M) \in \mathbb{R}^M$, and $\mathfrak{s} = (s_1, \ldots, s_M) \in \{\pm 1\}^M$. However, we have to restrict the analysis to speeds $c_j \neq 0$, since the function $\check{u}_0$, associated with the black soliton, vanishes at the origin.

Coming back to the original framework, we can define properly a corresponding sum of solitons $\boldsymbol{R}_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$, when the third component of $\boldsymbol{S}_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ does not reach the values ± 1. Due to the exponential decay of the functions v_c and w_c, this assumption is satisfied at least when the positions a_j are sufficiently separated, i.e. when the solitons are decoupled. In this case, the sum $\boldsymbol{R}_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ is given, up to a phase factor, by the expression

$$\boldsymbol{R}_{\mathfrak{c},\mathfrak{a},\mathfrak{s}} := \Big((1 - V_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}^2)^{\frac{1}{2}} \cos(\Phi_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}), (1 - V_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}^2)^{\frac{1}{2}} \sin(\Phi_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}), V_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}\Big),$$

where we have set

$$\Phi_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}(x) := \int_0^x W_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}(y)\,dy,$$

for any $x \in \mathbb{R}$. This definition presents the advantage to provide a quantity with values on the sphere $\mathbb{S}^2$. On the other hand, it is only defined under restrictive assumptions on the speeds c_j and positions a_j. Moreover, it does not take into account the geometric invariance with respect to rotations around the axis x_3.

3.2 Orbital stability in the energy space

In the sequel, our main results are proved in the hydrodynamical framework. We establish that, if the initial positions a_j^0 are well-separated and the initial speeds c_j^0 are ordered according to the initial positions a_j^0, then the solution corresponding to a chain of solitons at initial time, that is a perturbation of a sum of solitons $S_{\mathfrak{c}^0,\mathfrak{a}^0,\mathfrak{s}^0}$, is uniquely defined, and that it remains a chain of solitons for any positive time. We then rephrase this statement in the original framework.

Let us recall that Theorem 1.9 provides the existence and uniqueness of a continuous flow for (3.2) in the nonvanishing energy space $\mathcal{NV}(\mathbb{R})$. To our knowledge, the question of the global existence (in the hydrodynamical framework) of the local solution $\mathfrak{v}$ is open. In the sequel, we by-pass this difficulty using the stability of a well-prepared sum of solitons $\boldsymbol{S}_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$. Since the solitons in such a sum have exponential decay by (3.3), and are sufficiently well-separated, the sum $\boldsymbol{S}_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ belongs to $\mathcal{NV}(\mathbb{R})$. Invoking the Sobolev embedding theorem, this remains true for a small perturbation in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. As a consequence, the global existence for a well-prepared sum of solitons follows from its stability by applying a continuation argument.

Concerning the stability of sums of solitons, our main result is

Theorem 3.1 ([dLG15a]). *Let $\mathfrak{s}^* \in \{\pm 1\}^M$ and $\mathfrak{c}^* = (c_1^*, \ldots, c_M^*) \in (-1,1)^M$ such that*

$$c_1^* < \ldots < 0 < \ldots < c_M^*.$$

There exist positive numbers α^*, L^* and A^*, depending only on $\mathfrak{c}^*$ such that, if $\mathfrak{v}^0 \in \mathcal{NV}(\mathbb{R})$ satisfies the condition

$$\alpha^0 := \big\| \mathfrak{v}^0 - \boldsymbol{S}_{\mathfrak{c}^*,\mathfrak{a}^0,\mathfrak{s}^*} \big\|_{H^1 \times L^2} \leq \alpha^*,$$

for points $\mathfrak{a}^0 = (a_1^0, \ldots, a_M^0) \in \mathbb{R}^M$ such that

$$L^0 := \min \big\{ a_{j+1}^0 - a_j^0, 1 \leq j \leq M-1 \big\} \geq L^*,$$

then the solution $\mathfrak{v}$ to (3.2) with initial condition $\mathfrak{v}^0$ is globally well-defined on $\mathbb{R}_+$, and there exists a function $\boldsymbol{a} = (a_1, \ldots, a_M) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}^M)$ such that

$$\sum_{j=1}^M \big| a_j'(t) - c_j^* \big| \leq A^* \Big(\alpha^0 + \exp\Big(- \frac{\nu_{\mathfrak{c}^*} L^0}{65} \Big) \Big), \tag{3.4}$$

and

$$\big\| \mathfrak{v}(\cdot, t) - \boldsymbol{S}_{\mathfrak{c}^*,\boldsymbol{a}(t),\mathfrak{s}^*} \big\|_{H^1 \times L^2} \leq A^* \Big(\alpha^0 + \exp\Big(- \frac{\nu_{\mathfrak{c}^*} L^0}{65} \Big) \Big), \tag{3.5}$$

for any $t \in \mathbb{R}_+$.

Theorem 3.1 provides the orbital stability of well-prepared sums of solitons with different, nonzero speeds for positive time. The sums are well-prepared in the sense that their positions at initial time are well-separated and ordered according to their speeds (see condition (3.1)). As a consequence, the solitons are more and more separated along the LL flow (see estimate

(3.4)) and their interactions become weaker and weaker. The stability of the chain then results from the orbital stability of each single soliton in the chain.

As a matter of fact, the orbital stability of a single soliton appears as a special case of Theorem 3.1 when $M = 1$.

Corollary 3.2. *Let $s^* \in \{\pm 1\}$, $a^0 \in \mathbb{R}$ and $c^* \in (-1,0) \cup (0,1)$. There exist $\alpha^* > 0$ and $A^* > 0$, depending only on c^*, such that, if $\mathfrak{v}^0 \in \mathcal{NV}(\mathbb{R})$ satisfies the condition*

$$\alpha^0 := \big\|\mathfrak{v}^0 - \mathfrak{v}_{c^*,a^0,s^*}\big\|_{H^1 \times L^2} \leq \alpha^*,$$

then the solution $\mathfrak{v}$ to (3.2) with initial condition $\mathfrak{v}^0$ is globally well-defined on $\mathbb{R}$, and there exists a function $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that

$$\big|a'(t) - c^*\big| \leq A^* \alpha^0, \quad \textit{and } \big\|\mathfrak{v}(\cdot, t) - \mathfrak{v}_{c^*,a(t),s^*}\big\|_{H^1 \times L^2} \leq A^* \alpha^0,$$

for any $t \in \mathbb{R}$.

In this case, stability occurs for both positive and negative times due to the time reversibility of the LL equation. Time reversibility also provides the orbital stability of reversely well-prepared chains of solitons for negative time. The analysis of stability for both negative and positive time is more involved. It requires a deep understanding of the possible interactions between the solitons in the chain (see [MM09, MM11] for such an analysis in the context of the Korteweg-de Vries equation). This issue is of particular interest because of the existence of multisolitons.

Special chains of solitons are indeed provided by the exact multisolitons. However, there is a difficulty to define them properly in the hydrodynamical framework. As a matter of fact, multisolitons can reach the values ± 1 at some times. On the other hand, an arbitrary multisoliton becomes well-prepared for large time in the sense that the individual solitons are ordered according to their speeds and well-separated (see e.g. [BBI14, Section 10]).

If we consider a perturbation of an arbitrary multisoliton at initial time, our theorem does not guarantee that a perturbation of this multisoliton remains a perturbation of a multisoliton for large time. As a matter of fact, this property would follow from the continuity with respect to the initial datum of LL equation in the energy space, which remains, to our knowledge, an open question. Indeed, Corollary 1.10 only provides this continuity in the neighborhood of solutions $\boldsymbol{m}$, whose third component m_3 does not reach the value ± 1. As a consequence, Theorem 3.1 only shows the orbital stability of the multisolitons, which do not reach the values ± 1 for any positive time.

Let us go back now to the original formulation of the LL equation, so that we can rephrase the orbital stability of the sums of solitons follows.

Corollary 3.3. *Let $\mathfrak{s}^* \in \{\pm 1\}^M$ and $\mathfrak{c}^* = (c_1^*, \cdots, c_M^*) \in (-1,1)^M$, with $c_1^* < \cdots < 0 < \cdots < c_M^*$. Given any $\epsilon^* > 0$, there are $\rho^* > 0$ and $L^* > 0$ such that, if $\boldsymbol{m}^0 \in \mathcal{E}(\mathbb{R})$ satisfies the condition*

$$d_{\mathcal{E}}\big(\boldsymbol{m}^0, \boldsymbol{R}_{\mathfrak{c}^*,\mathfrak{a}^0,\mathfrak{s}^*}\big) \leq \rho^*, \tag{3.6}$$

for points $\mathfrak{a}^0 = (a_1^0, \ldots, a_M^0) \in \mathbb{R}^M$ *such that*

$$\min\big\{a_{j+1}^0 - a_j^0, 1 \leq j \leq M-1\big\} \geq L^*,$$

then the solution $\boldsymbol{m}$ *to* (3.1) *with initial datum* $\boldsymbol{m}^0$ *is globally well-defined on* $\mathbb{R}_+$. *Moreover, there exists a function* $\boldsymbol{a} = (a_1, \ldots, a_M) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}^M)$ *such that, setting*

$$I_1 := \Big(-\infty, \frac{a_1 + a_2}{2}\Big], \quad I_j := \Big[\frac{a_{j-1} + a_j}{2}, \frac{a_j + a_{j+1}}{2}\Big], \quad \text{and} \quad I_M := \Big[\frac{a_{M-1} + a_M}{2}, +\infty\Big),$$

for $2 \leq j \leq M-1$, *we have the estimates*

$$\sum_{j=1}^M \big|a_j'(t) - c_j^*\big| \leq \epsilon^*,$$

and

$$\sum_{j=1}^M \inf_{\theta_j \in \mathbb{R}} \Big\{\big|\check{m}(a_j(t), t) - \check{u}_{c_j^*, a_j(t), \theta_j, s_j^*}(a_j(t))\big| + \big\|\partial_x m - u'_{c_j^*, a_j(t), \theta_j, s_j^*}\big\|_{L^2(I_j)} + \big\|m_3 - \big[u_{c_j^*, a_j(t), \theta_j, s_j^*}\big]_3\big\|_{L^2(I_j)}\Big\} \leq \epsilon^*, \tag{3.7}$$

for any $t > 0$.

Corollary 3.3 only guarantees that a solution corresponding to a perturbation of a (well-prepared) sum of solitons $\boldsymbol{R}_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ at initial time splits into localized perturbations of M solitons for any time. In particular, the solution does not necessarily remain a perturbation of a sum of solitons $\boldsymbol{R}_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ for any time. This difficulty is related to the main obstacle when constructing a function $\boldsymbol{m}$ corresponding to a hydrodynamical pair $\mathfrak{v}$, which is a possible phase shift of the map $\check{m}$. In the construction of the sum $\boldsymbol{R}_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$, this phase shift is globally controlled. In contrast, the estimates into (3.5) do not seem to prevent a possible phase shift θ_j around each soliton in the hydrodynamical sum. This explains the difference between the controls in assumption (3.6) and in conclusion (3.7).

Observe also that we have no information on the dependence of the error ϵ^* on the numbers ρ^* and L^* in contrast with estimates (3.4) and (3.5) in Theorem 3.1. This is due to the property that the dependence of a function $\boldsymbol{m}$ with respect to the corresponding hydrodynamical pair $\mathfrak{v}$ is not a priori locally Lipschitz.

When $M = 1$, Corollary 3.3 states nothing more than the orbital stability of the solitons $\boldsymbol{u}_{c,a,\theta,s}$, with $c \neq 0$. Taking into account the time reversibility of the LL equation, we can indeed show

Corollary 3.4. *Let* $s^* \in \{\pm 1\}$, $a^0, \theta^0 \in \mathbb{R}$ *and* $c^* \in (-1, 0) \cup (0, 1)$. *Given any* $\epsilon^* > 0$, *there is* $\rho^* > 0$ *such that, if* $\boldsymbol{m}^0 \in \mathcal{E}(\mathbb{R})$ *satisfies the condition*

$$d_{\mathcal{E}}\big(\boldsymbol{m}^0, \boldsymbol{u}_{c^*, a^0, \theta^0, s^*}\big) \leq \rho^*,$$

then the solution $\boldsymbol{m}$ *to* (3.1) *with initial datum* $\boldsymbol{m}^0$ *is globally well-defined on* $\mathbb{R}_+$. *Moreover, there exists a function* $a \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$ *such that we have the estimates*

$$\big|a'(t) - c^*\big| \leq \epsilon^*,$$

and

$$\inf_{\theta\in\mathbb{R}} \Big\{ \big|\check{m}(a(t),t) - \check{u}_{c^*,a(t),\theta,s^*}(a(t))\big| + \big\|\partial_x m - u'_{c^*,a(t),\theta,s^*}\big\|_{L^2} + \big\|m_3 - \big[u_{c^*,a(t),\theta,s^*}\big]_3\big\|_{L^2} \Big\} \leq \epsilon^*,$$

for any $t \in \mathbb{R}$.

To our knowledge, the orbital stability of the soliton $\boldsymbol{u}_0$ remains an open question. In the context of the Gross–Pitaevskii equation, the orbital stability of the vanishing soliton (often called black soliton) was proved in [BGSS08a, GZ09]. Part of the analysis in this further context certainly extends to the soliton $\boldsymbol{u}_0$ of the LL equation.

In the rest of this section, we restrict our attention to the analysis of the stability of single solitons and sums of solitons in the hydrodynamical framework. In particular, we present below the main elements in the proof of Theorem 3.1. Before detailing this proof, we would like to underline that the arguments developed in the sequel do not make use of the inverse scattering transform. Instead, they rely on the Hamiltonian structure of the LL equation, in particular, on the conservation laws for the energy and momentum. As a consequence, our arguments can presumably be extended to non-integrable equations similar to the hydrodynamical LL equation.

Remark 3.5. In the isotropic case $\lambda_3 = 0$, there is no traveling-wave solution to (3.1) with nonzero speed and finite energy. However, breather-like solutions were found to exist in [LRT76a], and their numerical stability was investigated in [TW77]. In the easy-axis case, there are traveling-wave solutions (see e.g. [BL79]), but their third coordinate $m_3(x)$ converges to ± 1 as $|x| \to +\infty$. This prevents from invoking the hydrodynamical formulation, and thus from using the strategy developed below in order to prove their orbital stability.

3.2.1 Main elements in the proof of Theorem 3.1

Our strategy is reminiscent of the one developed to tackle the stability of well-prepared chains of solitons for the generalized Korteweg-de Vries equations [MMT02], the nonlinear Schrödinger equations [MMT06], or the Gross-Pitaevskii equation [BGS14].

A key ingredient in the proof is the minimizing nature of the soliton $\mathfrak{v}_c$, which can be constructed as the solution of the minimization problem

$$E(\mathfrak{v}_c) = \min \big\{ E(\mathfrak{v}), \mathfrak{v} \in \mathcal{NV}(\mathbb{R}) \text{ s.t. } P(\mathfrak{v}) = P(\mathfrak{v}_c) \big\}, \tag{3.8}$$

where we recall that the energy and the momentum are given by, for $\mathfrak{v} = (v, w)$,

$$E(\mathfrak{v}) = \frac{1}{2} \int_{\mathbb{R}} \Big(\frac{(v')^2}{1 - v^2} + \big(1 - v^2\big) w^2 + v^2 \Big), \quad \text{and } P(\mathfrak{v}) := \int_{\mathbb{R}} vw.$$

This characterization results from the compactness of the minimizing sequences for (3.8), and the classification of solitons in (3.3). The compactness of minimizing sequences can be proved following the arguments developed for a similar problem in the context of the Gross–Pitaevskii equation [BGS08a], that we will review and extend in chapter 5, in the context of the nonlocal Gross–Pitaevskii equation.

The Euler–Lagrange equation for (3.8) reduces to the identity

$$E'(\mathfrak{v}_c) = cP'(\mathfrak{v}_c), \tag{3.9}$$

where the speed c appears as the Lagrange multiplier of the minimization problem. The minimizing energy is equal to

$$E(\mathfrak{v}_c) = 2(1-c^2)^{\frac{1}{2}},$$

while the momentum of the soliton $\mathfrak{v}_c$ is given by

$$P(\mathfrak{v}_c) = 2\arctan\Big(\frac{(1-c^2)^{\frac{1}{2}}}{c}\Big), \tag{3.10}$$

when $c \neq 0$. An important consequence of formula (3.10) is the inequality

$$\frac{d}{dc}\Big(P(\mathfrak{v}_c)\Big) = -\frac{2}{(1-c^2)^{\frac{1}{2}}} < 0, \tag{3.11}$$

which is related to the Grillakis–Shatah–Strauss condition (see e.g. [GSS87]) for the orbital stability of a soliton. As a matter of fact, we can use inequality (3.11) to establish the coercivity of the quadratic form

$$Q_c := E''(\mathfrak{v}_c) - cP''(\mathfrak{v}_c),$$

under suitable orthogonality conditions. More precisely, we show

Proposition 3.6. *Let $c \in (-1,0) \cup (0,1)$. There exists $\Lambda_c > 0$, such that*

$$Q_c(\varepsilon) \geq \Lambda_c \|\varepsilon\|^2_{H^1\times L^2}, \tag{3.12}$$

for any pair $\varepsilon \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfying the two orthogonality conditions

$$\langle \partial_x \mathfrak{v}_c, \varepsilon\rangle_{L^2\times L^2} = \langle P'(\mathfrak{v}_c), \varepsilon\rangle_{L^2\times L^2} = 0. \tag{3.13}$$

Moreover, the map $c \mapsto \Lambda_c$ is uniformly bounded from below on any compact subset of $(-1,1)\setminus\{0\}$.

The first orthogonality condition in (3.13) originates in the invariance with respect to translations of (3.2). Due to this invariance, the pair $\partial_x \mathfrak{v}_c$ lies in the kernel of Q_c. The quadratic form Q_c also owns a unique negative direction, which is related to the constraint in (3.8). This direction is controlled by the second orthogonality condition in (3.13).

As a consequence of Proposition 3.6, the functional

$$F_c(\mathfrak{v}) := E(\mathfrak{v}) - cP(\mathfrak{v}),$$

controls any perturbation $\varepsilon = \mathfrak{v} - \mathfrak{v}_c$ satisfying the two orthogonality conditions in (3.13). More precisely, we derive from (3.9) and (3.12) that

$$F_c(\mathfrak{v}_c + \varepsilon) - F_c(\mathfrak{v}_c) \geq \Lambda_c \|\varepsilon\|_{H^1\times L^2}^2 + \mathcal{O}\big(\|\varepsilon\|_{H^1\times L^2}^3\big), \tag{3.14}$$

as $\|\varepsilon\|_{H^1\times L^2} \to 0$. Since the energy $E(\mathfrak{v})$ and the momentum $P(\mathfrak{v})$ are conserved along the flow, the left-hand side of (3.14) remains small for all time if it was small at the initial time. As a consequence of (3.14), the perturbation ε remains small for all time, which implies the stability of $\mathfrak{v}_c$.

The strategy for proving Theorem 3.1 consists in extending this argument to a sum of solitons. This requires to derive a coercivity inequality in the spirit of (3.14) for the perturbation of a sum of solitons. In a configuration where the solitons $\mathfrak{v}_{c_j,a_j,s_j}$ are sufficiently separated, a perturbation ε, which is localized around the position a_k, essentially interacts with the soliton $\mathfrak{v}_{c_k,a_k,s_k}$ due to the exponential decay of the solitons. In order to extend (3.14), it is necessary to impose that ε satisfies at least the orthogonality conditions in (3.13) for the soliton $\mathfrak{v}_{c_k,a_k,s_k}$. In particular, we cannot hope to extend (3.14) to a general perturbation ε without imposing the orthogonality conditions in (3.13) for all the solitons in the sum.

It turns out that this set of orthogonal conditions is sufficient to derive a coercivity inequality like (3.14) when the solitons in the sum are well-separated (see Proposition 3.8 below). Before addressing this question, we have to deal with the usual tool to impose orthogonality conditions, that is modulation parameters. Here again, we take advantage of the exponential decay of the solitons to check that modulating their speeds and positions is enough to get the necessary orthogonality conditions, at least when the solitons are well-separated.

More precisely, we now fix a set of speeds $\mathfrak{c}^* = (c_1^*, \ldots, c_M^*) \in (-1,1)^M$, with $c_j^* \neq 0$, and of orientations $\mathfrak{s}^* = (s_1^*, \ldots, s_n^*) \in \{\pm 1\}^M$ as in the statement of Theorem 3.1. Given a positive number L, we introduce the set of well-separated positions

$$\operatorname{Pos}(L) := \big\{\mathfrak{a} = (a_1, \ldots, a_M) \in \mathbb{R}^M, \text{ s.t. } a_{j+1} > a_j + L \text{ for } 1 \leq j \leq M-1\big\},$$

and we set

$$\mathcal{V}(\alpha, L) := \Big\{\mathfrak{v} = (v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ s.t. } \inf_{\mathfrak{a}\in \operatorname{Pos}(L)} \big\|\mathfrak{v} - \boldsymbol{S}_{\mathfrak{c}^*,\mathfrak{a},\mathfrak{s}^*}\big\|_{H^1\times L^2} < \alpha\Big\},$$

for any $\alpha > 0$. We also define

$$\mu_{\mathfrak{c}} := \min_{1\leq j\leq M} |c_j|, \quad \text{and} \quad \nu_{\mathfrak{c}} := \min_{1\leq j\leq M} \big(1 - c_j^2\big)^{\frac{1}{2}},$$

for any $\mathfrak{c} \in (-1,1)^M$. At least for α small enough and L sufficiently large, we show the existence of modulated speeds $\boldsymbol{c}(\mathfrak{v}) = (c_1(\mathfrak{v}), \ldots, c_M(\mathfrak{v}))$ and positions $\boldsymbol{a}(\mathfrak{v}) = (a_1(\mathfrak{v}), \ldots, a_M(\mathfrak{v}))$ such that any pair $\mathfrak{v} \in \mathcal{V}(\alpha, L)$ may be decomposed as $\mathfrak{v} = \boldsymbol{S}_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*} + \varepsilon$, with ε satisfying suitable orthogonality conditions.

Proposition 3.7. *There exist positive numbers α_1^* and L_1^*, depending only on $\mathfrak{c}^*$ and $\mathfrak{s}^*$, such that we have the following properties.*
(i) Any pair $\mathfrak{v} = (v, w) \in \mathcal{V}(\alpha_1^*, L_1^*)$ belongs to $\mathcal{NV}(\mathbb{R})$, with

$$1 - v^2 \geq \frac{1}{8}\mu_{\mathfrak{c}^*}^2.$$

(ii) There exist two maps $\boldsymbol{c} \in \mathcal{C}^1(\mathcal{V}(\alpha_1^*, L_1^*), (-1,1)^M)$ and $\boldsymbol{a} \in \mathcal{C}^1(\mathcal{V}(\alpha_1^*, L_1^*), \mathbb{R}^M)$ such that

$$\boldsymbol{\varepsilon} = \mathfrak{v} - \boldsymbol{S}_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*},$$

satisfies the orthogonality conditions

$$\big\langle \partial_x \mathfrak{v}_{c_j(\mathfrak{v}),a_j(\mathfrak{v}),s_j^*}, \boldsymbol{\varepsilon} \big\rangle_{L^2\times L^2} = \big\langle P'(\mathfrak{v}_{c_j(\mathfrak{v}),a_j(\mathfrak{v}),s_j^*}), \boldsymbol{\varepsilon} \big\rangle_{L^2\times L^2} = 0, \tag{3.15}$$

for any $1 \leq j \leq M$.
(iii) There exists a positive number A^*, depending only on $\mathfrak{c}^*$ and $\mathfrak{s}^*$, such that, if

$$\big\| \mathfrak{v} - \boldsymbol{S}_{\mathfrak{c}^*,\mathfrak{a}^*,\mathfrak{s}^*} \big\|_{H^1\times L^2} < \alpha,$$

for $\mathfrak{a}^* \in \mathrm{Pos}(L)$, with $L > L_1^*$ and $\alpha < \alpha_1^*$, then we have

$$\|\boldsymbol{\varepsilon}\|_{H^1\times L^2} + \sum_{j=1}^M \big| c_j(\mathfrak{v}) - c_j^* \big| + \sum_{j=1}^M \big| a_j(\mathfrak{v}) - a_j^* \big| \leq A^* \alpha, \tag{3.16}$$

as well as

$$\boldsymbol{a}(\mathfrak{v}) \in \mathrm{Pos}(L-1), \quad \mu_{\boldsymbol{c}(\mathfrak{v})} \geq \frac{1}{2}\mu_{\mathfrak{c}^*} \quad \text{and} \quad \nu_{\boldsymbol{c}(\mathfrak{v})} \geq \frac{1}{2}\nu_{\mathfrak{c}^*}.$$

The next ingredient in the proof is to check the persistence of a coercivity inequality like (3.14) for the perturbation $\boldsymbol{\varepsilon}$ in Proposition 3.7. Once again, we rely on the property that the solitons $\mathfrak{v}_j := \mathfrak{v}_{c_j(\mathfrak{v}),a_j(\mathfrak{v}),s_j^*}$ are well-separated and have exponential decay.

We indeed localize the perturbation $\boldsymbol{\varepsilon}$ around the position $a_j(\mathfrak{v})$ of each soliton $\mathfrak{v}_j$ by introducing cut-off functions, and we then control each localized perturbation using the coercivity of the quadratic form $Q_j = E''(\mathfrak{v}_j) - c_j(\mathfrak{v})P''(\mathfrak{v}_j)$ in (3.12). Such a control is allowed by the orthogonality conditions that we have imposed in (3.15). Collecting all the localized controls, we obtain a global bound on $\boldsymbol{\varepsilon}$, which is enough for our purpose.

More precisely, we consider a pair $\mathfrak{v} = (v, w) \in \mathcal{V}(\alpha_1^*, L_1^*)$, and we set

$$\boldsymbol{\varepsilon} = \mathfrak{v} - \boldsymbol{S}_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*},$$

as in Proposition 3.7, with $\boldsymbol{c}(\mathfrak{v}) = (c_1(\mathfrak{v}), \ldots, c_M(\mathfrak{v}))$ and $\boldsymbol{a}(\mathfrak{v}) = (a_1(\mathfrak{v}), \ldots, a_M(\mathfrak{v}))$. We next introduce the functions

$$\phi_j(x) := \begin{cases} 1 & \text{if } j = 1, \\ \frac{1}{2}\Big(1 + \tanh\Big(\frac{\nu_{\mathfrak{c}^*}}{16}\Big(x - \frac{a_{j-1}(\mathfrak{v}) + a_j(\mathfrak{v})}{2}\Big)\Big)\Big) & \text{if } 2 \leq j \leq M, \\ 0 & \text{if } j = M+1. \end{cases}$$

By construction, the maps $\phi_j - \phi_{j+1}$ are localized in a neighborhood of the soliton $\mathfrak{v}_j$. Moreover, they form a partition of unity since they satisfy the identity

$$\sum_{j=1}^{M} \big(\phi_j - \phi_{j+1}\big) = 1.$$

Setting

$$\mathcal{F}(\mathfrak{v}) := E(\mathfrak{v}) - \sum_{j=1}^{M} c_j^* P_j(\mathfrak{v}),$$

where

$$P_j(\mathfrak{v}) := \int_{\mathbb{R}} \big(\phi_j - \phi_{j+1}\big) vw,$$

and following the strategy described above, we prove that the functional $\mathcal{F}$ controls the perturbation $\boldsymbol{\varepsilon}$, up to small error terms.

Proposition 3.8. *There exist positive numbers $\alpha_2^* \leq \alpha_1^*$, $L_2^* \geq L_1^*$ and Λ^*, depending only on $\mathfrak{c}^*$ and $\mathfrak{s}^*$, such that $\mathfrak{v} = \boldsymbol{S}_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*} + \boldsymbol{\varepsilon} \in \mathcal{V}(\alpha_2^*, L)$, with $L \geq L_2^*$, satisfies the two inequalities*

$$\mathcal{F}(\mathfrak{v}) \geq \sum_{j=1}^{M} F_{c_j^*}(\mathfrak{v}_{c_j^*}) + \Lambda^* \big\|\boldsymbol{\varepsilon}\big\|_{H^1\times L^2}^2 + \mathcal{O}\Big(\sum_{j=1}^{M} |c_j(\mathfrak{v}) - c_j^*|^2\Big) + \mathcal{O}\Big(L \exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{16}\Big)\Big), \tag{3.17}$$

and

$$\mathcal{F}(\mathfrak{v}) \leq \sum_{j=1}^{M} F_{c_j^*}(\mathfrak{v}_{c_j^*}) + \mathcal{O}\Big(\big\|\boldsymbol{\varepsilon}\big\|_{H^1\times L^2}^2\Big) + \mathcal{O}\Big(\sum_{j=1}^{M} |c_j(\mathfrak{v}) - c_j^*|^2\Big) + \mathcal{O}\Big(L \exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{16}\Big)\Big).$$

Remark 3.9. Here as in the sequel, we have found convenient to use the notation $\mathcal{O}$ in order to simplify the presentation. By definition, we are allowed to substitute a quantity X by the notation $\mathcal{O}(Y)$ if and only if there exists a positive number A^*, depending only on $\mathfrak{c}^*$ and $\mathfrak{s}^*$, such that

$$|X| \leq A^* Y.$$

In order to establish the stability of a sum of solitons with respect to the LL flow, we now consider an initial datum $\mathfrak{v}^0 \in \mathcal{V}(\alpha/2, 2L)$, with $\alpha \leq \alpha_2^*$ and $L \geq L_2^*$. Invoking the continuity of the flow with respect to the initial datum (see Theorem 1.9), we can assume the existence of a positive number T such that

$$\mathfrak{v}(\cdot, t) \in \mathcal{V}(\alpha, L) \subset \mathcal{V}(\alpha_2^*, L_2^*),$$

for any $t \in [0, T]$. As a consequence, we can specialize the statements in Propositions 3.7 and 3.8 to the pair $\mathfrak{v}(\cdot, t)$. We define

$$\mathfrak{c}(t) := \boldsymbol{c}(\mathfrak{v}(\cdot, t)) := \big(c_1(t), \dots, c_M(t)\big), \quad \text{and} \quad \mathfrak{a}(t) := \boldsymbol{a}(\mathfrak{v}(\cdot, t)) := \big(a_1(t), \dots, a_M(t)\big),$$

as well as

$$\boldsymbol{\varepsilon}(\cdot,t) := \big(\varepsilon_1(\cdot,t),\varepsilon_2(\cdot,t)\big) = \mathfrak{v}(\cdot,t) - \boldsymbol{S}_{\mathfrak{c}(t),\mathfrak{a}(t),\mathfrak{s}^*}, \tag{3.18}$$

for any $t\in[0,T]$. In view of Proposition 3.7, we have

$$\|\boldsymbol{\varepsilon}(\cdot,t)\|_{H^1\times L^2} + \sum_{j=1}^M \big|c_j(t)-c_j^*\big| + \sum_{j=1}^M \big|a_j(t)-a_j^*\big| \leq A^*\alpha,$$

and

$$\boldsymbol{a}(t)\in \mathrm{Pos}(L-1), \quad \mu_{\boldsymbol{c}(t)} \geq \frac{1}{2}\mu_{\mathfrak{c}^*}, \quad \text{and} \quad \nu_{\boldsymbol{c}(t)} \geq \frac{1}{2}\nu_{\mathfrak{c}^*}.$$

Similarly, Proposition 3.8 provides

$$\begin{aligned}\mathcal{F}(t) := \mathcal{F}(\mathfrak{v}(\cdot,t)) \geq & \sum_{j=1}^M F_{c_j^*}(\mathfrak{v}_{c_j^*}) + \Lambda^*\big\|\boldsymbol{\varepsilon}(\cdot,t)\big\|_{H^1\times L^2}^2 + \mathcal{O}\Big(\sum_{j=1}^M |c_j(t)-c_j^*|^2\Big)\\ & + \mathcal{O}\Big(L\exp\Big(-\frac{\nu_{\mathfrak{c}^*}L}{16}\Big)\Big).\end{aligned}$$

Coming back to the strategy developed for the orbital stability of a single soliton (see the discussion after inequality (3.14)), we observe two major differences between the coercivity estimates (3.14) and (3.17). The first one lies in the two extra terms in the right-hand side of (3.17). There is no difficulty to control the second term, namely $\mathcal{O}(L\exp(-\nu_{\mathfrak{c}^*}L/16))$, since it becomes small when L is large enough. In contrast, we have to deal with the differences $|c_j(t)-c_j^*|^2$. In order to bound them, we rely on the equation satisfied by the perturbation $\boldsymbol{\varepsilon}$. Introducing identity (3.18) into (3.2) and using (3.9), we are led to the equations

$$\partial_t\varepsilon_1 = \sum_{j=1}^M \Big(\big(a_j'(t)-c_j(t)\big)\partial_x v_j - c_j'(t)\partial_c v_j\Big) + \partial_x\Big(\big((V+\varepsilon_1)^2-1\big)(W+\varepsilon_2) - \sum_{j=1}^M \big(v_j^2-1\big)w_j\Big),$$

and

$$\begin{aligned}\partial_t\varepsilon_2 = & \sum_{j=1}^M \Big(\big(a_j'(t)-c_j(t)\big)\partial_x w_j - c_j'(t)\partial_c w_j\Big) + \partial_{xx}\Big(\frac{\partial_x V+\partial_x\varepsilon_1}{1-(V+\varepsilon_1)^2} - \sum_{j=1}^M \frac{\partial_x v_j}{1-v_j^2}\Big)\\ & + \partial_x\Big(\big(V+\varepsilon_1\big)\big((W+\varepsilon_2)^2-1\big) - \frac{(V+\varepsilon_1)(\partial_x V+\partial_x\varepsilon_1)^2}{(1-(V+\varepsilon_1)^2)^2} - \sum_{j=1}^M\Big(v_j\big(w_j^2-1\big) - \frac{v_j(\partial_x v_j)^2}{(1-v_j^2)^2}\Big)\Big).\end{aligned}$$

Here, we have set $v_j(\cdot,t) := v_{c_j(t),a_j(t),s_j^*}(\cdot)$ and $w_j(\cdot,t) := w_{c_j(t),a_j(t),s_j^*}(\cdot)$ for any $1\leq j\leq M$, as well as

$$V(\cdot,t) = V_{\boldsymbol{c}(t),\boldsymbol{a}(t),\mathfrak{s}^*}(\cdot) = \sum_{j=1}^M v_j(\cdot,t), \quad \text{and} \quad W(\cdot,t) = W_{\boldsymbol{c}(t),\boldsymbol{a}(t),\mathfrak{s}^*}(\cdot) = \sum_{j=1}^M w_j(\cdot,t),$$

in order to simplify the notation. We next differentiate with respect to time the orthogonality conditions in (3.15) to derive bounds on the time derivatives $a_j'(t)$ and $c_j'(t)$ of the modulation parameters. This provides

Proposition 3.10. *There exist positive numbers $\alpha_3^* \leq \alpha_2^*$ and $L_3^* \geq L_2^*$, depending only on $\mathfrak{c}^*$ and $\mathfrak{s}^*$, such that, if $\alpha \leq \alpha_3^*$ and $L \geq L_3^*$, then the modulation functions $\boldsymbol{a}$ and $\boldsymbol{c}$ are of class $\mathcal{C}^1$ on $[0,T]$, and satisfy*

$$\sum_{j=1}^M \Big(\big| a_j'(t) - c_j(t) \big| + \big| c_j'(t) \big| \Big) = \mathcal{O}\Big(\big\| \boldsymbol{\varepsilon}(\cdot,t) \big\|_{H^1 \times L^2} \Big) + \mathcal{O}\Big(L \exp\Big(- \frac{\nu_{\mathfrak{c}^*} L}{2} \Big) \Big), \tag{3.19}$$

for any $t \in [0,T]$.

Combining Proposition 3.10 with the bounds in (3.16), we conclude that the evolution of the modulation parameters is essentially governed by the initial speeds of the solitons in the sum $\boldsymbol{S}_{\mathfrak{c}^*,\mathfrak{a}^*,\mathfrak{s}^*}$. In particular, when the speeds are well-ordered, that is when

$$c_1^* < \ldots < c_M^*, \tag{3.20}$$

the solitons in the sum $\boldsymbol{S}_{\mathfrak{c}(t),\mathfrak{a}(t),\mathfrak{s}^*}$ remain well-separated for any $t \in [0,T]$. More precisely, setting

$$\delta_{\mathfrak{c}^*} = \frac{1}{2} \min \big\{ c_{j+1}^* - c_j^*, 1 \leq j \leq M-1 \big\},$$

we can derive from (3.16), (3.19) and (3.20), for a possible further choice of the numbers α_3^* and L_3^*, the estimates

$$a_{j+1}(t) - a_j(t) > a_{j+1}(0) - a_j(0) + \delta_{\mathfrak{c}^*} t \geq L - 1 + \delta_{\mathfrak{c}^*} t,$$

and

$$a_j'(t)^2 \leq 1 - \frac{\nu_{\mathfrak{c}^*}^2}{4},$$

for any $t \in [0,T]$, when $\alpha \leq \alpha_3^*$ and $L \geq L_3^*$. In view of these bounds and the exponential decay of the solitons, the interactions between the solitons remain exponentially small for any $t \in [0,T]$.

A second difference between (3.14) and (3.17) lies in the fact that the left-hand side of (3.17) is not conserved along the (3.2) flow due to the presence of the cut-off function $\phi_j - \phi_{j+1}$ in the definition of P_j. As a consequence, we also have to control the evolution with respect to time of these quantities. We derive this control from the conservation law for the momentum, which may be written as

$$\partial_t \big(vw \big) = -\frac{1}{2} \partial_x \Big(v^2 + w^2 \big(1 - 3v^2 \big) + \frac{3 - v^2}{(1 - v^2)^2} (\partial_x v)^2 \Big) - \frac{1}{2} \partial_{xxx} \ln \big(1 - v^2 \big). \tag{3.21}$$

As a consequence of this equation, we obtain a monotonicity formula for a localized version of the momentum. More precisely, we set

$$R_j(t) = \int_{\mathbb{R}} \phi_j(\cdot,t) v(\cdot,t) w(\cdot,t),$$

for any $1 \leq j \leq M$. Using (3.21), we establish

Proposition 3.11. *There exist positive numbers $\alpha_4^* \leq \alpha_3^*$, $L_4^* \geq L_3^*$ and A_4^*, depending only on $\mathfrak{c}^*$ and $\mathfrak{s}^*$, such that, if $\alpha \leq \alpha_4^*$ and $L \geq L_4^*$, then the map R_j is of class $\mathcal{C}^1$ on $[0,T]$, and it satisfies*

$$R_j'(t) \geq -A_4^* \exp\Big(-\frac{\nu_{\mathfrak{c}^*}(L+\delta_{\mathfrak{c}^*}t)}{32}\Big), \tag{3.22}$$

for any $1 \leq j \leq M$ and any $t \in [0,T]$. In particular, the map $\mathcal{F}$ is of class $\mathcal{C}^1$ on $[0,T]$ and it satisfies

$$\mathcal{F}'(t) \leq \mathcal{O}\Big(\exp\Big(-\frac{\nu_{\mathfrak{c}^*}(L+\delta_{\mathfrak{c}^*}t)}{32}\Big)\Big), \tag{3.23}$$

for any $t \in [0,T]$.

Estimate (3.23) is enough to overcome the fact that the function $\mathcal{F}$ is not any longer conserved along time. We now have all the elements to complete the proof of Theorem 3.1 applying the strategy developed for the orbital stability of a single soliton.

3.3 Asymptotic stability

We consider now the long-time asymptotics of a solution to (3.1), with initial condition a perturbation of a soliton. We would like to determinate conditions such as the solutions converges to a (possible different) soliton.

Let us remark that the convergence as $t \to \infty$ cannot hold in the energy space. For instance, we could consider a solution $\mathfrak{v}$ to (3.2) with an initial condition $\mathfrak{v}^0 \in \mathcal{NV}(\mathbb{R})$, such that $\mathfrak{v}$ converges to a hydrodynamical soliton $\mathfrak{v}_c$ in the norm $\|\cdot\|_{H^1\times L^2}$, as $t \to \infty$. By the continuity of the energy and the momentum (with respect to this norm), we have

$$E\big(\mathfrak{v}(\cdot,t)\big) \to E(\mathfrak{v}_c) \quad \text{and} \quad P\big(\mathfrak{v}(\cdot,t)\big) \to P(\mathfrak{v}_c),$$

as $t \to \infty$. Since these quantities are conserved by the flow, we conclude that $E(\mathfrak{v}^0) = E(\mathfrak{v}_c)$ and $P(\mathfrak{v}^0) = P(\mathfrak{v}_c)$. Thus, the variational characterization of solitons implies that $\mathfrak{v}^0$ must be a soliton. Therefore, the only solutions that converge (in energy norm) to a soliton as $t \to \infty$, are the solitons.

In conclusion, to establish the asymptotic stability, we need to weak the notion of convergence. Indeed, using the weak convergence in the space $\mathcal{NV}(\mathbb{R})$, Bahri [Bah16] proved the asymptotic stability of solitons for the hydrodynamical formulation of LL.

Theorem 3.12 ([Bah16])**.** *Let $c \in (-1,1)$, $c \neq 0$. There is $\alpha^* > 0$ such that, if the initial condition $\mathfrak{v}^0 \in \mathcal{NV}(\mathbb{R})$ satisfies that*

$$\big\|\mathfrak{v}^0 - \mathfrak{v}_c\big\|_{H^1\times L^2} < \alpha^*,$$

then there exist a unique global associated solution $\mathfrak{v} \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$ to (3.1), $c^* \in (-1,0) \cup (0,1)$ and $a \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ such that, as $t \to \infty$,

$$\mathfrak{v}(\cdot + a(t), t)\big) \rightharpoonup \mathfrak{v}_{c^*} \quad \text{in} \quad H^1(\mathbb{R}) \times L^2(\mathbb{R}), \qquad \textit{and} \qquad a'(t) \to c^*.$$

This kind of result has been previously established for the solitons of other equations such as the Korteweg-de Vries [MM01], the Benjamin-Bona-Mahony [Dik05], the Benjamin-Ono [KM09], and the Gross-Pitaevskii [BGS15a, GS15a] equations.

The weak convergence in Theorem 3.12 can probably be improved. Indeed, Martel and Merle [MM05, MM08b, MM08a] proved the asymptotic stability of solitons of the KdV equation, establishing a locally (strong) convergence in the energy space. It is possible that a similar result can be shown for the asymptotic stability of hydrodynamical solitons of the LL equation satisfy a similar, i.e. a strong convergence in a norm of the type $H^1([-R(t), R(t)]) \times L^2([-R(t), R(t)])$, where $R(t)$ is a linear function of time.

Finally, we remark that Theorem 3.12 provides the weak convergence towards a soliton, but this long-time dynamics need to take into account the geometric in variances of the problem, i.e. the translations. This is precisely the role of the parameter $a(t)$, whose derivative converges to the speed of the limit soliton $\mathfrak{v}_{c^*}$. In this fashion, the solution propagates with the same speed as the limit soliton, as t goes to infinity, as expected.

Going back to the original framework for the LL equation, Bahri [Bah16] also obtained

Corollary 3.13 ([Bah16])**.** *Let $c \in (-1, 1)$, $c \neq 0$. There exists $\delta^* > 0$ such that, if $\boldsymbol{m}^0 \in \mathcal{E}(\mathbb{R})$ satisfies*

$$d_{\mathcal{E}}(\boldsymbol{m}^0, \boldsymbol{m}_c) < \delta^*,$$

then there are $c^* \in (-1, 0) \cup (0, 1)$, and two functions $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and $\theta \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that the associated solution $\boldsymbol{m}$ to (3.1) *with initial condition $\boldsymbol{m}^0$ satisfies*

$$\partial_x \boldsymbol{m} - \boldsymbol{m}'_{c^*, a(t), \theta(t)} \rightharpoonup 0, \quad \text{and} \quad m_3 - [\boldsymbol{m}_{c^*, a(t), \theta(t)}]_3 \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}),$$

as $t \to \infty$. Also, a and θ satisfy, as $t \to \infty$,

$$a'(t) \to c^* \quad \text{and} \quad \theta'(t) \to 0.$$

This result is a almost a direct consequence of Theorem 3.12, and the only difficulty is to handle the invariance by rotations around the axis x_3 of the original equation (3.1). The phase θ allows to by-pass this difficulty. Let us remark that as $t \to \infty$, the derivative of the phase converge to 0, and thus the orientation of the solution in the plan $x_3 = 0$ is asymptotically fixed. Let us also remark that, by the Rellich–Kondrachov theorem, the weak convergence in Theorem 3.13 provides a local uniform convergence towards the limit soliton.

The proof of Theorems 3.12 and 3.13 are based on an approach developed by Martel and Merle for the KdV equation (cf. [MM01, MM05, Mar06, MM08b, MM08a]). Their strategy can be decomposed in three steps, that we would explain in our context, i.e. in the hydrodynamical setting.

The orbital stability provided by Theorem 3.1 guarantees that a solution $\mathfrak{v}$ to (3.1), with initial condition $\mathfrak{v}^0$ close enough to a soliton $\mathfrak{v}_c$, remains in a neighborhood of the orbit of the soliton. In particular, the solution $\mathfrak{v}$ is bounded in the nonvanishing space $\mathcal{NV}(\mathbb{R})$ for any $t \geq 0$. It is then possible to construct a sequence of times (t_n), with $t_n \to \infty$, and a limit function $\mathfrak{v}_*^0 \in \mathcal{NV}(\mathbb{R})$, such that, up to a subsequence,

$$\mathfrak{v}(\cdot, t_n) \rightharpoonup \mathfrak{v}_*^0 \quad \text{dans} \quad H^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

as $n \to \infty$. In addition, $\mathfrak{v}_*^0$ remains close to the orbit of the soliton $\mathfrak{v}_c$. Moreover, the solution $\mathfrak{v}_*$ to (3.2) with initial condition $\mathfrak{v}_*^0$ is global, and is also close to this orbit. We point out that is also necessary to introduce a modulation parameter due to the invariance by translation, but we will omit it for the sake of clarity.

We need to prove that the limit profile $\mathfrak{v}_*^0$, and the associated solution $\mathfrak{v}_*$, are indeed solitons. Thus, the second step is to study the regularity and decay properties of $\mathfrak{v}_*$. To this end, it is useful to establish the weak continuity of the flow of the hydrodynamical equation with respect to the initial condition, which imply that the solution $\mathfrak{v}$ converges to $\mathfrak{v}_*$ as follows: for any $t \in \mathbb{R}$ (fixed),

$$\mathfrak{v}(\cdot, t_n + t) \rightharpoonup \mathfrak{v}_*(\cdot, t) \quad \text{in} \quad H^1(\mathbb{R}) \times L^2(\mathbb{R}).$$

Using also the monotonicity formula for the momentum in Proposition 3.11, from this convergence it is possible to deduce that $\mathfrak{v}_*$ is localized in space, uniformly in time. Moreover, (3.22) also implies that $\mathfrak{v}_*$ has an exponential decay in space, uniformly in time. Thus, using the Kato smoothing effect that gives regularizing properties of the Schrödinger-type equations, it follows that $\mathfrak{v}_*$ is of class $\mathcal{C}^\infty$ on $\mathbb{R} \times \mathbb{R}$, and that all its derivatives also decay in space, uniformly in time (see e.g. [KPV03, BGS15a] for more details).

The third step is to show that in the neighborhood of a soliton, the only solutions to (3.2) having this behavior are the solitons. This rigidity property follows from a Liouville type theorem. The proof of this theorem requires another monotonicity formula, and it is the most difficult part of the argument. We refer to [Bah16] for more details.

By refining the approach described above, Bahri [Bah18] also established the asymptotic stability for initial data close to a sum of solitons, that are as usual well-prepared according to their speeds and have sufficiently separated initial positions. We conclude this section with the following result that states that the associated solution converges weakly to one of the solitons in the sum, when it is translated to the center of this soliton.

Theorem 3.14 ([Bah18]). *Let $\mathfrak{s}^* \in \{\pm 1\}^M$ and $\mathfrak{c}^* = (c_1^*, \dots, c_M^*) \in (-1,1)^M$ such that*

$$c_1^* < \dots < 0 < \dots < c_M^*.$$

There exist $\alpha^* > 0$, $L^* > 0$ and $A^* > 0$, depending only on $\mathfrak{c}^*$ such that, if $\mathfrak{v}^0 \in \mathcal{NV}(\mathbb{R})$ satisfies

$$\alpha^0 := \big\| \mathfrak{v}^0 - \sum_{j=1}^M \mathfrak{v}_{c_j^*, a_j^0, s_j} \big\|_{H^1 \times L^2} \leq \alpha^*,$$

for points $\mathfrak{a}^0 = (a_1^0, \dots, a_M^0) \in \mathbb{R}^M$ such that

$$L^0 := \min \big\{ a_{j+1}^0 - a_j^0, 1 \leq j \leq M-1 \big\} \geq L^*,$$

then the global solution $\mathfrak{v}$ to (3.2) *with initial condition $\mathfrak{v}^0$ satisfies the following properties. There exist $\mathfrak{a} = (a_1, \dots, a_M) \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R}^M)$, $\mathfrak{c} = (c_1, \dots, c_M) \in \mathcal{C}^1(\mathbb{R}^+, ((-1,1) \setminus \{0\})^M)$, and $\mathfrak{c}^\infty = (c_1^\infty, \dots, c_M^\infty) \in ((-1,1) \setminus \{0\})^M$, such that, for all $j \in \{1, \dots M\}$,*

$$\mathfrak{v}(\cdot + a_j(t), t) - \sum_{j=1}^M s_j \mathfrak{v}_{c_k(t)}(\cdot + a_j(t) - a_k(t)) \rightharpoonup 0, \qquad \text{in } H^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

and

$$c_j(t) \to c_j^\infty, \quad a_j'(t) \to c_j^\infty, \quad ast \to \infty.$$

In particular, as $t \to \infty$*,*

$$\mathfrak{v}(\cdot + a_j(t), t) - s_j \mathfrak{v}_{c_k(t)}(\cdot) \rightharpoonup 0 \quad \text{in } H^1(\mathbb{R}) \times L^2(\mathbb{R}).$$

The proof of this theorem relies on the strategy developed by Martel, Merle and Tsai in [MMT02] for the KdV equation. Let us also remark that the locally strong asymptotic stability result for multisolitons in [MMT02] is stronger than the statement in Theorem 3.14 with $M = 2$. Indeed, the proof in [MMT02] is based on a monotonicity argument for the localized energy. It is an open problem if this kind of argument can be adapted to the study of the LL equation, or more generally, if it possible to get a locally strong asymptotic stability result. We refer the interested reader to [KMMn17] for a survey on asymptotic stability of other dispersive equations and wave models.

Chapter 4

Self-similar solutions for the LLG equation

In this chapter we will study the dissipative LLG equation (14) in the isotropic case, i.e.

$$\partial_t \boldsymbol{m} = \beta \boldsymbol{m} \times \Delta \boldsymbol{m} - \alpha \boldsymbol{m} \times (\boldsymbol{m} \times \Delta \boldsymbol{m}), \tag{4.1}$$

with $\alpha > 0$. We will focus on the existence of self-similar solutions and provide their asymptotics in dimension $N = 1$. We also analyze the qualitative and quantitative effect of the damping α on the dynamical behavior of these self-similar solutions.

As we will see, these kinds of solutions do not belong to classical Sobolev spaces, and we cannot invoke the Cauchy theory developed in Chapter 1 to give a meaning to their stability. Therefore, we will provide a well-posedness result in a more general framework related to the BMO space to give some stability results. We point out that the proof of the well-posedness result uses the parabolic behavior of the equation in presence of damping, and cannot be applied for the pure dispersive equation (i.e. $\alpha = 0$) analyzed in previous chapters.

The results present in this chapter are based on joint works with S. Gutiérrez [dLG15b, dLG19, dLG20].

4.1 Self-similar solutions

A natural question, that has been proven relevant for understanding the global behavior of solutions and formation of singularities, is whether or not there exist solutions which are invariant under scalings of the equation. In the case of equation (4.1), it is straightforward to see that it is invariant under the following scaling: If $\boldsymbol{m}$ is a solution of (4.1), then $\boldsymbol{m}_\lambda(t, x) = \boldsymbol{m}(\lambda x, \lambda^2 t)$ is also a solution, for any $\lambda > 0$. Associated with this invariance, a solution $\boldsymbol{m}$ of (4.1) defined for $I = \mathbb{R}^+$ or $I = \mathbb{R}^-$ is called *self-similar* if it is invariant under rescaling, that is

$$\boldsymbol{m}(x, t) = \boldsymbol{m}(\lambda x, \lambda^2 t), \quad \forall \lambda > 0, \quad \forall x \in \mathbb{R}^N, \quad \forall t \in I.$$

Setting $T \in \mathbb{R}$ and performing a translation in time, this definition leads to two types of self-similar solutions: A forward self-similar solution, or *expander*, is a solution of the form

$$\boldsymbol{m}(x,t) = \boldsymbol{f}\left(\frac{x}{\sqrt{t-T}}\right) \qquad \text{for} \quad (x,t) \in \mathbb{R}^N \times (T,\infty),$$

and a backward self-similar solution, or *shrinker*, is a solution of the form

$$\boldsymbol{m}(x,t) = \boldsymbol{f}\left(\frac{x}{\sqrt{T-t}}\right) \qquad \text{for} \quad (x,t) \in \mathbb{R}^N \times (-\infty,T),$$

for certain profile $\boldsymbol{f} : \mathbb{R}^N \longrightarrow \mathbb{S}^2$. Expanders evolve from a singular value at time T, while shrinkers evolve towards a singular value at time T.

Self-similar solutions have brought a lot of attention in the study on nonlinear PDEs because they can provide some important information about the dynamics of the equation. While expanders are related to nonuniqueness phenomena, resolution of singularities and long time description of solutions, shrinkers are often related to phenomena of singularity formation (see e.g. [GGS10, EF09]). On the other hand, the construction and understanding of the dynamics and properties of self-similar solutions also provide an idea of which are the natural spaces to develop a well-posedness theory, that captures these very often physically relevant structures. Examples of equations for which self-similar solutions have been considered, and a substantial work around these types of solutions has been done, include among others the Navier–Stokes equation, semilinear parabolic equations, and geometric flows such as Yang-Mills, mean curvature flow and harmonic map flow. We refer to [JST18, QS07, Ilm98, Str88] and the references therein for more details.

Most of the works in the literature related to the study of self-similar solutions to the LLG equation are confined to the heat flow for harmonic maps equation, i.e. $\alpha = 1$. In this setting, the main works on the subject restrict the analysis to corotational maps taking values in $\mathbb{S}^d$, which reduces the analysis of (HFHM) to the study of a second order real-valued ODE. Then tools such as the maximum principle or the shooting method can be used to show the existence of solutions. We refer to [Fan99, Gas02, GR11, BD18, BW15, BB11, GGM17] and the references therein for more details on such results for maps taking values in $\mathbb{S}^d$, with $d \geq 3$. Recently, Deruelle and Lamm [DL] have studied the Cauchy problem for the harmonic map heat flow with initial data $\boldsymbol{m}^0 : \mathbb{R}^N \to \mathbb{S}^d$, with $N \geq 3$ and $d \geq 2$, where $\boldsymbol{m}^0$ is Lipschitz 0-homogeneous function, homotopic to a constant, which implies the existence of expanders coming out of $\boldsymbol{m}^0$.

When $0 < \alpha \leq 1$, we recently established the existence of self-similar expanders for the LLG equation in [dLG19]. This result is a consequence of a well-possedness theorem for the LLG equation considering an initial data $\boldsymbol{m}^0 : \mathbb{R}^N \to \mathbb{S}^2$ in the space BMO of functions of bounded mean oscillation. Notice that this result includes in particular the case of the harmonic map heat flow. We will explain more precisely this result in Section 4.3.

As seen before, in absence of damping ($\alpha = 0$), (4.1) reduces to the Schrödinger map equation (4), which is reversible in time, so that the notions of expanders and shrinkers coincide. For this equation, Germain, Shatah and Zeng [GSZ10] established the existence of (k-equivariant) self-similar profiles $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{S}^2$.

In the one-dimensional case, when $\alpha = 0$, (4) is closely related Localized Induction Approximation (LIA), and self-similar profiles $\boldsymbol{f} : \mathbb{R} \to \mathbb{S}^2$ were obtained and analyzed in [GRV03, GV04, LRT76b, But88]. In the context of LIA, self-similar solutions constitute a family of smooth solutions that develop a singularity in the shape of a corner in finite time. For further work related to these solutions, including the study of the continuation of these solutions after the blow-up time and their stability, we refer to the reader to [BV18, BV09]. At the level of the Schrödinger map equation, these self-similar solutions provide examples of smooth solutions that develop a jump singularity in finite time.

In the next section we explain how to construct the family of expanders profiles for $\alpha \in [0, 1]$, and provide their analytical study [dLG15b]. In Section 4.3, we also discuss the Cauchy problem associated with these solutions and their stability [dLG19]. Finally, in Section 4.5, we construct and analyze the family of shrinkers profiles [dLG20]

4.2 Expanders in dimension one

We consider in this section equation (4.1) in dimension $N = 1$, and $\alpha \in [0, 1]$, in order to include both the damped and undamped cases. We seek self-similar solutions of the form

$$\boldsymbol{m}(x,t) = \mathfrak{m}\left(\frac{x}{\sqrt{t}}\right), \qquad x \in \mathbb{R},\ t > 0. \tag{4.2}$$

and we will say that $\mathfrak{m}$ is the *profile* of the solution $\boldsymbol{m}$. Observe that if $\boldsymbol{m}$ is a solution to (4.1) given by a smooth profile $\mathfrak{m}$ as in (4.2), then $\mathfrak{m}$ solves the following system of ODEs

$$-\frac{x\mathfrak{m}'}{2} = \beta\mathfrak{m} \times \mathfrak{m}'' - \alpha\mathfrak{m} \times (\mathfrak{m} \times \mathfrak{m}''), \quad \text{ on } \mathbb{R}, \tag{4.3}$$

which recasts as

$$\alpha\mathfrak{m}'' + \alpha|\mathfrak{m}'|^2\mathfrak{m} + \beta(\mathfrak{m} \times \mathfrak{m}')' + \frac{x\mathfrak{m}'}{2} = 0, \quad \text{ on } \mathbb{R}, \tag{4.4}$$

due to the fact that $\mathfrak{m}$ takes values in $\mathbb{S}^2$. Thus, we can give a weak formulation to this equation, in the sense that $\mathfrak{m} \in H^1_{\rm loc}(\mathbb{R}; \mathbb{S}^2)$ solves the system

$$\int_{\mathbb{R}} \mathcal{A}(\mathfrak{m}(x))\mathfrak{m}'(x) \cdot \varphi'(x) = \int_{\mathbb{R}} G(x, \mathfrak{m}, \mathfrak{m}')\varphi(x), \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}),$$

with

$$\mathcal{A}(\boldsymbol{u}) = \begin{pmatrix} \alpha & -\beta u_3 & \beta u_2 \\ \beta u_3 & \alpha & -\beta u_1 \\ -\beta u_2 & \beta u_1 & \alpha, \end{pmatrix} \quad \text{ and } \quad G(x, \boldsymbol{u}, \boldsymbol{p}) = \begin{pmatrix} \alpha u_1|\boldsymbol{p}|^2 - \frac{xp_1}{2} \\ \alpha u_2|\boldsymbol{p}|^2 - \frac{xp_2}{2} \\ \alpha u_3|\boldsymbol{p}|^2 - \frac{xp_3}{2} \end{pmatrix},$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{p} = (p_1, p_2, p_3)$. If $\alpha > 0$, this system is uniformly elliptic, since

$$\mathcal{A}(\boldsymbol{u})\boldsymbol{\xi} \cdot \boldsymbol{\xi} = \alpha|\boldsymbol{\xi}|^2, \quad \text{ for all } \boldsymbol{\xi}, \boldsymbol{u} \in \mathbb{R}^3.$$

We can then invoke the regularity theory for quasilinear elliptic systems (see [LU68, Gia83]), to verify that the solutions are smooth. In the limit case $\alpha = 0$, we can show directly that the solutions are also smooth. Most importantly, we have the following theorem that provides a rigidity result concerning the possible solutions to (4.4): The modulus of the gradient of any solution *must* be $ce^{-\alpha x^2/4}$, for some $c \geq 0$.

Theorem 4.1. *Let $\alpha \in [0,1]$. Assume that $\mathfrak{m} \in H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{S}^2)$ is a weak solution to (4.4). Then $\mathfrak{m}$ belongs to $\mathcal{C}^\infty(\mathbb{R};\mathbb{S}^2)$ and there exists $c \geq 0$ such that $|\mathfrak{m}'(x)| = ce^{-\alpha x^2/4}$, for all $x \in \mathbb{R}$.*

In the limit cases $\alpha = 1$ and $\alpha = 0$, it is possible to find explicit solutions to (4.4), as we will see later on. However, this seems unlikely in the case $\alpha \in (0,1)$, and even the existence of such solutions is not clear. We proceed now to give a way of to establish the *existence* of solutions satisfying the condition $|\mathfrak{m}'(x)| = ce^{-\alpha x^2/4}$, for any $c > 0$ (notice that case $c = 0$ is trivial), and any $\alpha \in [0,1]$.

The idea is to look for $\mathfrak{m}$ as the tangent vector to a curve in $\mathbb{R}^3$, so we first recall some facts about curves in the space. Given $\mathfrak{m} : \mathbb{R} \to \mathbb{S}^2$ a smooth function, we can define the curve

$$\boldsymbol{X}_{\mathfrak{m}}(x) = \int_0^x \mathfrak{m}(s)ds, \tag{4.5}$$

so that $\boldsymbol{X}_{\mathfrak{m}}$ is smooth, parametrized by arclenght, and its tangent vector is $\mathfrak{m}$. In addition, if $|\mathfrak{m}'|$ does not vanish on $\mathbb{R}$, we can define the normal vector $\mathfrak{n}(x) = \mathfrak{m}'(x)/|\mathfrak{m}'(x)|$ and the binormal vector $\mathfrak{b}(x) = \mathfrak{m}(x) \times \mathfrak{n}(x)$. Moreover, we can define the curvature and torsion of $\boldsymbol{X}_{\mathfrak{m}}$ as $k(x) = |\mathfrak{m}'(x)|$ and $\tau(x) = -\mathfrak{b}'(x) \cdot \mathfrak{n}(x)$. Since $|\mathfrak{m}(x)|^2 = 1$, for all $x \in \mathbb{R}$, we have that $\mathfrak{m}(x) \cdot \mathfrak{n}(x) = 0$, for all $x \in \mathbb{R}$, that the vectors $\{\mathfrak{m}, \mathfrak{n}, \mathfrak{b}\}$ are orthonormal and it is standard to check that they satisfy the Serret–Frenet system

$$\begin{aligned} \mathfrak{m}' &= k\mathfrak{n}, \\ \mathfrak{n}' &= -k\mathfrak{m} + \tau\mathfrak{b}, \\ \mathfrak{b}' &= -\tau\mathfrak{n}. \end{aligned} \tag{4.6}$$

Let us apply this method to find a solution to (4.3). We define $\boldsymbol{X}_{\mathfrak{m}}$ as in (4.5), and we remark that equation (4.3) rewrites in terms of $\{\mathfrak{m}, \mathfrak{n}, \mathfrak{b}\}$ as

$$-\frac{x}{2}k\mathfrak{n} = \beta(k'\mathfrak{b} - \tau k\mathfrak{n}) - \alpha(-k'\mathfrak{n} - k\tau\mathfrak{b}).$$

Therefore, from the orthogonality of the vectors $\mathfrak{n}$ and $\mathfrak{b}$, we conclude that the curvature and torsion of $\boldsymbol{X}_{\mathfrak{m}}$ are solutions of the equations

$$-\frac{x}{2}k = \alpha k' - \beta\tau k \qquad \text{and} \qquad \beta k' + \alpha k\tau = 0,$$

that is

$$k(x) = ce^{-\frac{\alpha x^2}{4}} \qquad \text{and} \qquad \tau(x) = \frac{\beta x}{2}, \tag{4.7}$$

for some $c \geq 0$. Of course, the fact that $k(x) = ce^{-\alpha x^2/4}$ is in agreement with the fact that we must have $|\mathfrak{m}'(x)| = ce^{-\alpha x^2/4}$.

Now, given $\alpha \in [0,1]$ and $c > 0$, consider the Serret–Frenet system (4.6) with curvature and torsion function given by (4.7) and initial conditions

$$\mathfrak{m}(0) = (1,0,0), \quad \mathfrak{n}(0) = (0,1,0), \quad \mathfrak{b}(0) = (0,0,1). \tag{4.8}$$

Then, by standard ODE theory, there exists a unique global solution $\{\mathfrak{m}_{c,\alpha}, \mathfrak{n}_{c,\alpha}, \mathfrak{b}_{c,\alpha}\}$ in $(\mathcal{C}^\infty(\mathbb{R};\mathbb{S}^2))^3$, and these vectors are orthonormal. Also, it is straightforward to verify that $\mathfrak{m}_{c,\alpha}$ is a solution to (4.3) (and to (4.4)) satisfying $|\mathfrak{m}'_{c,\alpha}(x)| = ce^{-\alpha x^2/4}$.

The above argument provides the existence of solutions in the statement of Theorem 4.1. Finally, using the uniqueness of the Cauchy–Lipshitz theorem and the Serret–Frenet system, it is simple to show the uniqueness of such solutions, up to rotations.

Theorem 4.2 ([dLG20])**.** *The set of nonconstant solutions to* (4.3) *is* $\{\mathcal{R}\mathfrak{m}_{c,\alpha} : c > 0, \mathcal{R} \in SO(3)\}$*, where* $SO(3)$ *is the group of rotations about the origin preserving orientations.*

The above proposition reduces the study of self-expanders to the understanding of the family of self-expanders associated with the profiles $\{\mathfrak{m}_{c,\alpha}\}_{c,\alpha}$. The next result summarize the properties of these solutions.

Theorem 4.3 ([dLG15b])**.** *Let* $\alpha \in [0,1]$*,* $c \geq 0$ *and* $\mathfrak{m}_{c,\alpha}$ *be the solution of the Serret–Frenet system* (4.6) *with curvature and torsion given by* (4.7) *and initial conditions* (4.8)*. Let*

$$\boldsymbol{m}_{c,\alpha}(x,t) = \mathfrak{m}_{c,\alpha}\left(\frac{x}{\sqrt{t}}\right), \qquad \textit{for} \quad (x,t) \in \mathbb{R} \times (0,\infty).$$

Then the following statements hold.

(i) *The function* $\boldsymbol{m}_{c,\alpha}$ *is a* $\mathcal{C}^\infty(\mathbb{R};\mathbb{S}^2)$*-solution of* (4.1) *on* $\mathbb{R} \times (0,\infty)$*.*

(ii) *There exist unitary vectors* $\boldsymbol{A}^{\pm}_{c,\alpha} = (A^{\pm}_{j,c,\alpha})_{j=1}^3 \in \mathbb{S}^2$ *such that the following pointwise convergence holds when* t *goes to zero:*

$$\lim_{t\to 0^+} \boldsymbol{m}_{c,\alpha}(x,t) = \begin{cases} \boldsymbol{A}^+_{c,\alpha}, & \textit{if } s > 0, \\ \boldsymbol{A}^-_{c,\alpha}, & \textit{if } s < 0, \end{cases}$$

where $\boldsymbol{A}^-_{c,\alpha} = (A^+_{1,c,\alpha}, -A^+_{2,c,\alpha}, -A^+_{3,c,\alpha})$*.*

(iii) *Moreover, there exists a constant* $C(c,\alpha,p)$ *such that for all* $t > 0$

$$\|\boldsymbol{m}_{c,\alpha}(\cdot,t) - \boldsymbol{A}^+_{c,\alpha}\chi_{\mathbb{R}^+}(\cdot) - \boldsymbol{A}^-_{c,\alpha}\chi_{\mathbb{R}^-}(\cdot)\|_{L^p(\mathbb{R})} \leq C(c,\alpha,p)t^{\frac{1}{2p}}, \tag{4.9}$$

for all $p \in (1,\infty)$*. In addition, if* $\alpha > 0$*,* (4.9) *also holds for* $p = 1$*. Here,* χ_E *denotes the characteristic function of a set* E*.*

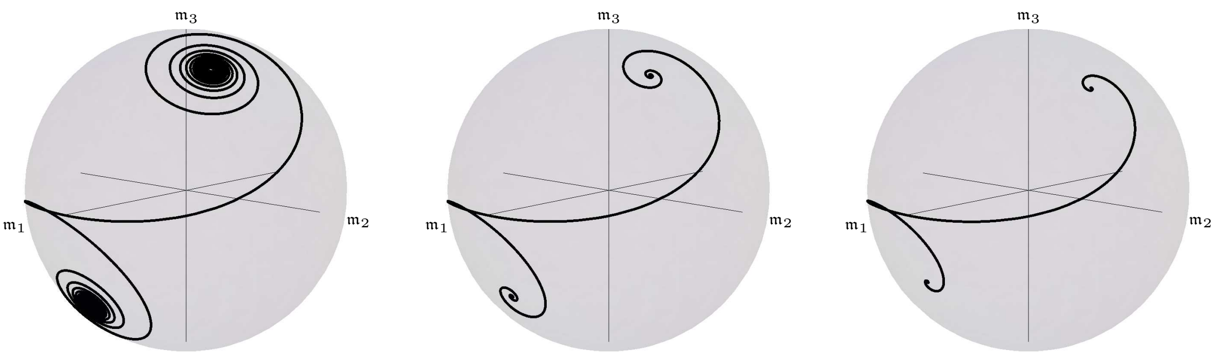
(iv) *For* $t > 0$ *and* $x \in \mathbb{R}$*, the derivative in space satisfies*

$$|\partial_x \boldsymbol{m}_{c,\alpha}(x,t)| = \frac{c}{\sqrt{t}}e^{-\frac{\alpha x^2}{4t}}. \tag{4.10}$$

Let us point out that the case $c = 0$ is trivial and corresponds to the constant solution

$$\boldsymbol{m}_{c,\alpha}(x,t) = \boldsymbol{m}\left(\frac{s}{\sqrt{t}}\right) = (1,0,0), \qquad \forall \alpha \in [0,1].$$

The graphics in Figure 4.1 depict the profile $\mathfrak{m}_{c,\alpha}$ for fixed $c = 0.8$ and the values of $\alpha = 0.01$, $\alpha = 0.2$, and $\alpha = 0.4$. In particular, it can be observed how the convergence of $\mathfrak{m}_{c,\alpha}$ to $\boldsymbol{A}^{\pm}_{c,\alpha}$ is accelerated by the diffusion α.



(a) $\alpha = 0.01$ (b) $\alpha = 0.2$ (c) $\alpha = 0.4$

Figure 4.1: The profile $\mathfrak{m}_{c,\alpha}$ for $c = 0.8$ and different values of α.

Notice that the initial condition

$$\boldsymbol{m}^0_{c,\alpha} := \boldsymbol{A}^+_{c,\alpha}\chi_{\mathbb{R}^+} + \boldsymbol{A}^-_{c,\alpha}\chi_{\mathbb{R}^-}, \tag{4.11}$$

has a jump singularity at the point $x = 0$ whenever the vectors $\boldsymbol{A}^+_{c,\alpha}$ and $\boldsymbol{A}^-_{c,\alpha}$ satisfy

$$\boldsymbol{A}^+_{c,\alpha} \neq \boldsymbol{A}^-_{c,\alpha}.$$

In this situation (and we will be able to prove analytically that this is the case, at least for certain ranges of the parameters α and c, see Proposition 4.7 below), Theorem 4.3 provides a bi-parametric family of global smooth solutions of (4.1) associated to a discontinuous singular initial data (jump-singularity).

As has been already mentioned, in the absence of damping ($\alpha = 0$), singular self-similar solutions of the Schrödinger map equation were previously obtained in [GRV03], [LRT76b] and [But88]. In this framework, Theorem 4.3 establishes the persistence of a jump singularity for self-similar solutions in the presence of dissipation.

When $\alpha = 0$, the stability of the self-similar solutions was considered in a series of papers by Banica and Vega [BV09, BV12, BV13]. The stability in the case $\alpha > 0$ is a natural question that we will discuss in Section 4.3.

Some further remarks on the results stated in Theorem 4.3 are in order. First, the total energy $E(t)$ of the $\boldsymbol{m}_{c,\alpha}(x,t)$ for $\alpha > 0$, is given by

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} |\partial_x \boldsymbol{m}_{c,\alpha}(x,t)|^2\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{c}{\sqrt{t}}e^{-\frac{\alpha x^2}{4t}}\right)^2 dx = c^2\sqrt{\frac{\pi}{\alpha t}}, \quad t > 0. \tag{4.12}$$

It is evident from (4.12) that the total energy at the initial time $t = 0$ is infinite, while the total energy becomes finite for all positive times, showing the dissipation of energy in the system in the presence of damping.

Secondly, it is also important to remark that in the setting of Schrödinger equations, for fixed $\alpha \in [0,1]$ and $c > 0$, the solution $\boldsymbol{m}_{c,\alpha}$ of (4.1) established in Theorem 4.3 is associated through the Hasimoto transformation (19) with the filament function

$$u_{c,\alpha}(x,t) = \frac{c}{\sqrt{t}}e^{(-\alpha+i\beta)\frac{x^2}{4t}}, \tag{4.13}$$

which solves

$$i\partial_t u + (\beta - i\alpha)\partial_{xx}u + \frac{u}{2}\left(\beta|u|^2 + 2\alpha\int_0^x \operatorname{Im}(\bar{u}\partial_x u) - A(t)\right) = 0, \quad \text{with} \quad A(t) = \frac{\beta c^2}{t}, \tag{4.14}$$

and is such that at as $t \to 0^+$

$$u_{c,\alpha}(x,0) := \lim_{t\to 0^+} u_{c,\alpha}(x,t) = 2c\sqrt{\pi(\alpha+i\beta)}\delta_0, \quad \text{in } S'(\mathbb{R}).$$

Here δ_0 denotes the Dirac delta function at the origin, and $\sqrt{z}$ denotes the square root of a complex number z such that $\operatorname{Im}(\sqrt{z}) > 0$.

Notice that the solution $u_{c,\alpha}(x,t)$ is very rough at initial time, and in particular $u_{c,\alpha}(x,0)$ does not belong to the Sobolev class $H^s(\mathbb{R})$, for any $s \geq 0$. Therefore, the standard arguments (i.e. a Picard iteration scheme based on Strichartz estimates and Sobolev–Bourgain spaces) cannot be applied, at least not in a straightforward way, to study the local well-posedness of the initial value problem for the Schrödinger equations (4.14). The existence of solutions to equation (4.14) associated with an initial data proportional to a Dirac delta opens the question of developing a well-posedness theory for Schrödinger equations of the type considered here to include initial data of infinite energy. In the case $\alpha = 0$, $A(t) = 0$ and when the initial condition is proportional to the Dirac delta, Kenig, Ponce and Vega [KPV01] proved that the Cauchy problem for (4.14) is ill-posed due to some oscillations. Moreover, even after removing these oscillations, Banica and Vega [BV09] showed that the associated equation (4.14) (with $\alpha = 0$ and $A(t) = c^2/t$) is still ill-posed. This question was also addressed by Vargas and Vega in [VV01] and Grünrock in [Grü05] for other types of initial data of infinite energy (see also [BV08] for a related problem), but we are not aware of any results in this setting when $\alpha > 0$ (see [GD08] for related well-posedness results in the case $\alpha > 0$ for initial data in Sobolev spaces of positive index). Notice that when $\alpha > 0$, the solution (4.13) has infinite energy at the initial time, however the energy becomes finite for any $t > 0$. Moreover, as a consequence of the exponential decay in the space variable when $\alpha > 0$, $u_{c,\alpha}(t) \in H^m(\mathbb{R})$, for all $t > 0$ and $m \in \mathbb{N}$. Hence, these solutions do not fit into the usual functional framework for solutions of the Schrödinger equations (4.14).

4.2.1 Asymptotics for the profile

In this section we study the qualitative and quantitative effect of the damping α and the parameter c on the dynamical behavior of the family $(\boldsymbol{m}_{c,\alpha})_{c,\alpha}$ of self-similar solutions of (4.1) found in Theorem 4.3. Precisely, in an attempt to fully understand the regularization of the solution at positive times close to the initial time $t = 0$, and to understand how the presence of damping affects the dynamical behavior of these self-similar solutions, we aim to give answers to the following questions:

Q1: Can we obtain a more precise behavior of the solutions $\boldsymbol{m}_{c,\alpha}(x,t)$ at positive times t close to zero?

Q2: Can we understand the limiting vectors $\boldsymbol{A}^{\pm}_{c,\alpha}$ in terms of the parameters c and α?

In order to address our first question, we observe that, due to the self-similar nature of these solutions, the behavior of the family of solutions $\boldsymbol{m}_{c,\alpha}$ at positive times close to the initial time $t = 0$ is directly related to the study of the asymptotics of the associated profile $\mathfrak{m}_{c,\alpha}(x)$ for large values of $|x|$. In addition, the symmetries of $\mathfrak{m}_{c,\alpha}$ (see Theorem 4.4 below) allow to reduce ourselves to obtain the behavior of the profile as $x \to \infty$. The precise asymptotics of the profile is given in the following theorem.

Theorem 4.4 ([dLG15b]). *Let $\alpha \in [0,1]$ and $c > 0$.*

(i) *(Symmetries). The components of $\mathfrak{m}_{c,\alpha}$, $\mathfrak{n}_{c,\alpha}$ and $\mathfrak{b}_{c,\alpha}$ satisfy respectively that*

- *$\mathfrak{m}_{1,c,\alpha}$ is an even function, and $\mathfrak{m}_{j,c,\alpha}$ is an odd function for $j \in \{2,3\}$.*
- *$\mathfrak{n}_{1,c,\alpha}$ and $\mathfrak{b}_{1,c,\alpha}$ are odd functions, while $\mathfrak{n}_{j,c,\alpha}$ and $\mathfrak{b}_{j,c,\alpha}$ are even functions for $j \in \{2,3\}$.*

(ii) *(Asymptotics). There exist a unit vector $\boldsymbol{A}^{+}_{c,\alpha} \in \mathbb{S}^2$, and $\boldsymbol{B}^{+}_{c,\alpha} \in \mathbb{R}^3$ such that the following asymptotics hold for all $s \geq s_0 = 4\sqrt{8+c^2}$:*

$$\begin{aligned}\mathfrak{m}_{c,\alpha}(s) =& \boldsymbol{A}^{+}_{c,\alpha} - \frac{2c}{s}\boldsymbol{B}^{+}_{c,\alpha} e^{-\alpha s^2/4}(\alpha \sin(\boldsymbol{\phi}_{c,\alpha}(s)) + \beta \cos(\boldsymbol{\phi}_{c,\alpha}(s))) \\ & - \frac{2c^2}{s^2}\boldsymbol{A}^{+}_{c,\alpha} e^{-\alpha s^2/2} + O\left(\frac{e^{-\alpha s^2/4}}{s^3}\right),\end{aligned} \tag{4.15}$$

$$\mathfrak{n}_{c,\alpha}(s) = \boldsymbol{B}^{+}_{c,\alpha}\sin(\boldsymbol{\phi}_{c,\alpha}(s)) + \frac{2c}{s}\boldsymbol{A}^{+}_{c,\alpha}\alpha e^{-\alpha s^2/4} + O\left(\frac{e^{-\alpha s^2/4}}{s^2}\right),$$

$$\mathfrak{b}_{c,\alpha}(s) = \boldsymbol{B}^{+}_{c,\alpha}\cos(\boldsymbol{\phi}_{c,\alpha}(s)) + \frac{2c}{s}\boldsymbol{A}^{+}_{c,\alpha}\beta e^{-\alpha s^2/4} + O\left(\frac{e^{-\alpha s^2/4}}{s^2}\right). \tag{4.16}$$

Here, $\sin(\boldsymbol{\phi}_{c,\alpha})$ and $\cos(\boldsymbol{\phi}_{c,\alpha})$ are understood acting on each of the components of $\boldsymbol{\phi}_{c,\alpha} = (\phi_{1,c\alpha}, \phi_{2,c,\alpha}, \phi_{3,c,\alpha})$, with

$$\phi_{j,c,\alpha}(s) = a_{j,\alpha,c} + \beta \int_{s_0^2/4}^{s^2/4} \sqrt{1 + c^2\frac{e^{-2\alpha\sigma}}{\sigma}}\, d\sigma, \quad j \in \{1,2,3\}, \tag{4.17}$$

for some constants $a_{1,\alpha,c}, a_{2,\alpha,c}, a_{1,\alpha,c} \in [0,2\pi)$, *and the vector* $\boldsymbol{B}^+_{c,\alpha}$ *is given in terms of* $\boldsymbol{A}^+_{c,\alpha} = (A^+_{j,c,\alpha})_{j=1}^3$ *by*

$$\boldsymbol{B}^+_{c,\alpha} = ((1-(A^+_{1,c,\alpha})^2)^{1/2}, (1-(A^+_{2,c,\alpha})^2)^{1/2}, (1-(A^+_{3,c,\alpha})^2)^{1/2}).$$

As we will see later, the convergence and rate of convergence of the solutions $\boldsymbol{m}_{c,\alpha}$ established in parts (ii) and (iii) of Theorem 4.3, will be obtained as a consequence of the more refined asymptotic analysis of the associated profile given in Theorem 4.4.

With regard to the asymptotics of the profile established in part (ii) of Theorem 4.4, it is important to mention that the errors in the asymptotics in Theorem 4.4-(ii) depend only on c. In other words, the bounds for the errors terms are independent of $\alpha \in [0,1]$. More precisely, we use the notation $O(f(s))$ to denote a function for which exists a constant $C(c) > 0$ depending on c, but independent on α, such that

$$|O(f(s))| \leq C(c)|f(s)|, \quad \text{for all } s \geq s_0.$$

At first glance, one might think that the term $-2c^2\boldsymbol{A}^+_{c,\alpha}e^{-\alpha s^2/2}/s^2$ in (4.15) could be included in the error term $O(e^{-\alpha s^2/4}/s^3)$. However, we cannot do this because

$$\frac{e^{-\alpha s^2/2}}{s^2} > \frac{e^{-\alpha s^2/4}}{s^3}, \qquad \text{for all } 2 \leq s \leq \left(\frac{2}{3\alpha}\right)^{1/2}, \ \alpha \in (0,1/8], \tag{4.18}$$

and in our notation the big-O must be independent of α. (The exact interval where the inequality in (4.18) holds can be determined using the so-called Lambert W function.)

When $\alpha = 1$ (so $\beta = 0$), we can solve explicitly the Serret–Frenet system, to obtain

$$\begin{aligned}
\mathfrak{m}_{c,1}(s) &= (\cos(c\,\mathrm{Erf}(s)), \sin(c\,\mathrm{Erf}(s)), 0),\\
\mathfrak{n}_{c,1}(s) &= -(\sin(c\,\mathrm{Erf}(s)), \cos(c\,\mathrm{Erf}(s)), 0),\\
\mathfrak{b}_{c,1}(s) &= (0,0,1),
\end{aligned}$$

for all $s \in \mathbb{R}$, where Erf is the non-normalized error function

$$\mathrm{Erf}(s) = \int_0^s e^{-\sigma^2/4}\, d\sigma.$$

In particular, the limiting vectors $\boldsymbol{A}^+_{c,1}$ and $\boldsymbol{A}^-_{c,1}$ in Theorem 4.4 are given in terms of c by

$$\boldsymbol{A}^\pm_{c,1} = (\cos(c\sqrt{\pi}), \pm\sin(c\sqrt{\pi}), 0), \tag{4.19}$$

and we have

$$\begin{aligned}
\mathfrak{m}_{c,1}(s) &= \boldsymbol{A}^+_{c,1} - \frac{2c}{s}\boldsymbol{B}^+_{c,1}e^{-s^2/4}\sin(\boldsymbol{a}_{c,1}) - \frac{2c^2}{s^2}\boldsymbol{A}^+_{c,1}e^{-s^2/2} + O\left(\frac{e^{-s^2/4}}{s^3}\right),\\
\mathfrak{n}_{c,1}(s) &= \boldsymbol{B}^+_{c,1}\sin(\boldsymbol{a}_{c,1}) + \frac{2c}{s}\boldsymbol{A}^+_{c,1}e^{-s^2/4} - \frac{2c^2}{s^2}\boldsymbol{B}^+_{c,1}e^{-s^2/2}\sin(\boldsymbol{a}_{c,1}) + O\left(\frac{e^{-s^2/4}}{s^3}\right),\\
\mathfrak{b}_{c,1}(s) &= \boldsymbol{B}^+_{c,1}\cos(\boldsymbol{a}_{c,1}),
\end{aligned}$$

where

$$\boldsymbol{B}^+_{c,1} = (|\sin(c\sqrt{\pi})|, |\cos(c\sqrt{\pi})|, 1),$$

and $\boldsymbol{a}_{c,1} = (a_{1,c,1}, a_{2,c,1}, 0)$, with

$$a_{1,c,1} = \begin{cases} \frac{3\pi}{2}, & \text{if } \sin(c\sqrt{\pi}) \geq 0, \\ \frac{\pi}{2}, & \text{if } \sin(c\sqrt{\pi}) < 0, \end{cases} \qquad a_{2,c,1} = \begin{cases} \frac{\pi}{2}, & \text{if } \cos(c\sqrt{\pi}) \geq 0, \\ \frac{3\pi}{2}, & \text{if } \cos(c\sqrt{\pi}) < 0. \end{cases}$$

When $\alpha = 0$, the solution of (4.6) can be solved explicitly in terms of parabolic cylinder functions or confluent hypergeometric functions (see [GL19, AS64]). Another analytical approach using Fourier analysis techniques has been taken in [GRV03], leading to the asymptotics

$$\begin{aligned} \mathfrak{m}_{c,0}(s) =& \boldsymbol{A}^+_{c,0} - \frac{2c}{s}\boldsymbol{B}^+_{c,0} + O\left(1/s^2\right), \\ \mathfrak{n}_{c,\alpha}(s) =& \boldsymbol{B}^+_{c,0} \sin(\boldsymbol{\psi}_c(s)) + O\left(1/s\right), \\ \mathfrak{b}_{c,\alpha}(s) =& \boldsymbol{B}^+_{c,0} \cos(\boldsymbol{\psi}_c(s)) + O\left(1/s\right), \end{aligned} \tag{4.20}$$

where

$$\boldsymbol{\psi}_{j,c} = b_{j,c} + \frac{s^2}{4} + c^2 \ln(s), \tag{4.21}$$

for some constants $b_{j,c}$, $j \in \{1,2,3\}$. Moreover, denoting by Γ the Euler Gamma function, $\boldsymbol{A}^+_{c,0}$ is explicitly given by

$$\begin{aligned} A^+_{1,c,0} &= e^{-\frac{\pi c^2}{2}}, \\ A^+_{2,c,0} &= 1 - \frac{e^{-\frac{\pi c^2}{4}}}{8\pi} \sinh(\pi c^2/2) |c\Gamma(ic^2/4) + 2e^{i\pi/4}\Gamma(1/2 + ic^2/4)|^2, \\ A^+_{3,c,0} &= 1 - \frac{e^{-\frac{\pi c^2}{4}}}{8\pi} \sinh(\pi c^2/2) |c\Gamma(ic^2/4) - 2e^{-i\pi/4}\Gamma(1/2 + ic^2/4)|^2. \end{aligned} \tag{4.22}$$

On the other hand, for fixed $j \in \{1,2,3\}$, we can write $\boldsymbol{\phi}_{c,\alpha}$ in (4.17) as

$$\phi_{j,c,\alpha}(s) = a_{j,c,\alpha} + \frac{s^2}{4} + c^2 \ln(s) + C(c) + O\left(1/s^2\right),$$

so that we recover the logarithmic contribution in the oscillation in (4.21).

When $\alpha > 0$, $\boldsymbol{\phi}_{c,\alpha}$ behaves like

$$\phi_{j,c,\alpha}(s) = a_{j,c,\alpha} + \frac{\beta s^2}{4} + C(\alpha, c) + O\left(\frac{e^{-\alpha s^2/2}}{\alpha s^2}\right),$$

and there is no logarithmic correction in the oscillations in the presence of damping.

Consequently, the phase function $\boldsymbol{\phi}_{c,\alpha}$ captures the different nature of the oscillatory character of the solutions in both the absence and the presence of damping in the system of equations.

It can be seen that the terms $\boldsymbol{A}^+_{c,\alpha}$, $\boldsymbol{B}^+_{c,\alpha} = (B^+_j)^3_{j=1}$, $B^+_j \sin(a_j)$ and $B^+_j \cos(a_j)$, $j \in \{1,2,3\}$, and the error terms in Theorem 4.4-(ii) depend continuously on $\alpha \in [0,1]$. Therefore, the asymptotics (4.15)–(4.16) show how the profile $\mathfrak{m}_{c,\alpha}$ converges to $\mathfrak{m}_{c,0}$ as $\alpha \to 0^+$ and to $\mathfrak{m}_{c,1}$ as $\alpha \to 1^-$. In particular, we recover the asymptotics in (4.20).

Finally, the amplitude of the leading order term controlling the wave-like behavior of the solution $\mathfrak{m}_{c,\alpha}(s)$ around $\boldsymbol{A}^\pm_{c,\alpha}$ for values of s sufficiently large is of the order $c\, e^{-\alpha s^2/4}/s$, from which one observes how the convergence of the solution to its limiting values $\boldsymbol{A}^\pm_{c,\alpha}$ is accelerated in the presence of damping in the system, as depicted in Figure 4.1.

4.2.2 Dependence on the parameters

Let us discuss now some results answering the second of our questions Q2. Bearing in mind that $\boldsymbol{A}^-_{c,\alpha}$ is expressed in terms of the coordinates of $\boldsymbol{A}^+_{c,\alpha}$ as $\boldsymbol{A}^-_{c,\alpha} = (A^+_{1,c,\alpha}, -A^+_{2,c,\alpha}, -A^+_{3,c,\alpha})$ (see part (ii) of Theorem 4.3), we only need to focus on $\boldsymbol{A}^+_{c,\alpha}$. When $\alpha = 1$ or $\alpha = 0$, the vector $\boldsymbol{A}^+_{c,\alpha}$ is explicitly given in terms of the parameter c in formulas (4.19) and (4.22), respectively. However, when $\alpha \in (0,1)$, we do not have explicit expressions for these vectors. The purpose of Theorems 4.5 and 4.6 below is therefore to establish the dependence of $\boldsymbol{A}^\pm_{c,\alpha}$ with respect to the parameters α and c. Theorem 4.5 provides the behavior of the limiting vector $\boldsymbol{A}^+_{c,\alpha}$ for a fixed value of $\alpha \in (0,1]$ and "small" values of $c > 0$, while Theorem 4.6 states the behavior of $\boldsymbol{A}^+_{c,\alpha}$ for fixed $c > 0$, and α close to the limiting values $\alpha = 0$ and $\alpha = 1$.

Theorem 4.5 ([dLG15b])**.** *Let $\alpha \in [0,1]$, $c > 0$, and $\boldsymbol{A}^+_{c,\alpha} = (A^+_{j,c,\alpha})^3_{j=1}$ be the unit vector given in Theorem 4.4. Then $\boldsymbol{A}^+_{c,\alpha}$ is a continuous function of $c \geq 0$. Moreover, if $\alpha \in (0,1]$ the following inequalities hold true.*

$$|A^+_{1,c,\alpha} - 1| \leq \frac{c^2\pi}{\alpha}\left(1 + \frac{c^2\pi}{8\alpha}\right),$$

$$\left|A^+_{2,c,\alpha} - c\frac{\sqrt{\pi(1+\alpha)}}{\sqrt{2}}\right| \leq \frac{c^2\pi}{4} + \frac{c^2\pi}{\alpha\sqrt{2}}\left(1 + \frac{c^2\pi}{8} + c\frac{\sqrt{\pi(1+\alpha)}}{2\sqrt{2}}\right) + \left(\frac{c^2\pi}{2\sqrt{2}\alpha}\right)^2,$$

$$\left|A^+_{3,c,\alpha} - c\frac{\sqrt{\pi(1-\alpha)}}{\sqrt{2}}\right| \leq \frac{c^2\pi}{4} + \frac{c^2\pi}{\alpha\sqrt{2}}\left(1 + \frac{c^2\pi}{8} + c\frac{\sqrt{\pi(1-\alpha)}}{2\sqrt{2}}\right) + \left(\frac{c^2\pi}{2\sqrt{2}\alpha}\right)^2.$$

In particular $\boldsymbol{A}^+_{c,\alpha} \to (1,0,0)$ as $c \to 0^+$, for any $\alpha \in [0,1]$.

Theorem 4.6. *Let $c > 0$, $\alpha \in [0,1]$ and $\boldsymbol{A}^+_{c,\alpha}$ be the unit vector given in Theorem 4.4. Then $\boldsymbol{A}^+_{c,\alpha}$ is a continuous function of α in $[0,1]$, and the following inequalities hold true.*

$$|\boldsymbol{A}^+_{c,\alpha} - \boldsymbol{A}^+_{c,0}| \leq C(c)\sqrt{\alpha}|\ln(\alpha)|, \quad \textit{for all} \ \ \alpha \in (0,1/2],$$
$$|\boldsymbol{A}^+_{c,\alpha} - \boldsymbol{A}^+_{c,1}| \leq C(c)\sqrt{1-\alpha}, \quad \textit{for all} \ \ \alpha \in [1/2,1].$$

Here, $C(c)$ is a positive constant depending on c but independent of α.

As a by-product of Theorems 4.5 and 4.6, we obtain the following proposition which asserts that the solutions $\boldsymbol{m}_{c,\alpha}$ of the LLG equation found in Theorem 4.3 are indeed associated to a discontinuous initial data at least for certain ranges of α and c.

Proposition 4.7. *With the same notation as in Theorems 4.3 and 4.4, the following statements hold.*

(i) For fixed $\alpha \in (0,1]$ there exists $c^* > 0$ depending on α such that

$$\boldsymbol{A}^+_{c,\alpha} \neq \boldsymbol{A}^-_{c,\alpha} \qquad \textit{for all} \;\; c \in (0, c^*).$$

(ii) For fixed $c > 0$, there exists $\alpha_0^* > 0$ small enough such that

$$\boldsymbol{A}^+_{c,\alpha} \neq \boldsymbol{A}^-_{c,\alpha} \qquad \textit{for all} \;\; \alpha \in (0, \alpha_0^*).$$

(iii) For fixed $0 < c \neq k\sqrt{\pi}$ with $k \in \mathbb{N}$, there exists $\alpha_1^* > 0$ with $1 - \alpha_1^* > 0$ small enough such that

$$\boldsymbol{A}^+_{c,\alpha} \neq \boldsymbol{A}^-_{c,\alpha} \qquad \textit{for all} \;\; \alpha \in (\alpha_1^*, 1).$$

Remark 4.8. From (4.22) we get $\boldsymbol{A}^+_{c,0} \neq \boldsymbol{A}^-_{c,0}$ for all $c > 0$. Based on the numerical results in [dLG15b], we conjecture that $\boldsymbol{A}^+_{c,\alpha} \neq \boldsymbol{A}^-_{c,\alpha}$ for all $\alpha \in (0,1)$ and $c > 0$.

In the next section we provide a framework to study the Cauchy problem associated with LLG, when considering a perturbation of a self-similar solution $\boldsymbol{m}_{c\alpha}$, as initial condition. In addition, as an application of previous results, we will see that when α is close to 1, there are multiple smooth solutions associated with the same initial condition.

4.2.3 Elements of the proofs of Theorems 4.3 and 4.4

Classical changes of variables from the differential geometry of curves allow us to reduce the nine equations in the Serret–Frenet system into three complex-valued second order equations (see [Dar93, Str50, Lam80]). These changes of variables are related to the stereographic projection; this approach was used in [GRV03]. However, their choice of stereographic projection has a singularity at the origin, which leads to an indetermination of the initial conditions of some of the new variables. For this reason, we consider in the following lemma a stereographic projection that is compatible with the initial conditions (4.8).

Lemma 4.9. *Let $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{b})$ be a solution of the Serret–Frenet equations* (4.6) *with positive curvature k and torsion τ. Then, for each $j \in \{1,2,3\}$, the function*

$$f_j(s) = e^{\frac{1}{2}\int_0^s k(\sigma)\eta_j(\sigma)\,d\sigma}, \qquad \textit{with} \qquad \eta_j(s) = \frac{(n_j(s) + i b_j(s))}{1 + m_j(s)},$$

solves the equation

$$f_j''(s) + \left(i\tau(s) - \frac{k'(s)}{k(s)}\right) f_j'(s) + \frac{k^2(s)}{4} f_j(s) = 0,$$

with initial conditions

$$f_j(0) = 1, \quad f_j'(0) = \frac{k(0)(n_j(0) + ib_j(0))}{2(1 + m_j(0))}.$$

Moreover, the coordinates of $\boldsymbol{m}$*,* $\boldsymbol{n}$ *and* $\boldsymbol{b}$ *are given in terms of* f_j *and* f_j' *by*

$$m_j(s) = 2\left(1 + \frac{4}{k(s)^2}\left|\frac{f_j'(s)}{f_j(s)}\right|^2\right)^{-1} - 1, \quad n_j(s) + ib_j(s) = \frac{4f_j'(s)}{k(s)f_j(s)}\left(1 + \frac{4}{k(s)^2}\left|\frac{f_j'(s)}{f_j(s)}\right|^2\right)^{-1}.$$

The above relations are valid at least as long as $m_j > -1$ *and* $|f_j| > 0$*.*

Going back to our problem, Lemma 4.9 reduces the analysis of the solution $\{\mathfrak{m}_{c,\alpha}, \mathfrak{n}_{c,\alpha}, \mathfrak{b}_{c,\alpha}\}$ of the Serret–Frenet system (16) with curvature and torsion given by (4.7) and initial conditions (4.8), to the study of three solutions to second order differential equation[1]

$$f''(s) + \frac{s}{2}(\alpha + i\beta)f'(s) + \frac{c^2}{4}e^{-\alpha s^2/2}f(s) = 0, \tag{4.23}$$

associated with three initial conditions, that we denote f_1, f_2 and f_3. More precisely,

- For $(m_1, n_1, b_1) = (1, 0, 0)$ the associated initial condition for f_1 is
$$f_1(0) = 1, \quad f_1'(0) = 0. \tag{4.24}$$
- For $(m_2, n_2, b_2) = (0, 1, 0)$ the associated initial condition for f_2 is
$$f_2(0) = 1, \quad f_2'(0) = c/2. \tag{4.25}$$
- For $(m_3, n_3, b_3) = (0, 0, 1)$ the associated initial condition forf_3 is
$$f_3(0) = 1, \quad f_3'(0) = ic/2. \tag{4.26}$$

It is important to notice that, by multiplying (4.23) by $e^{\frac{\alpha s^2}{2}}\bar{f}'$ and taking the real part, it is easy to see that

$$\frac{d}{ds}\left[\frac{1}{2}\left(e^{\frac{\alpha s^2}{2}}|f'|^2 + \frac{c^2}{4}|f|^2\right)\right] = 0.$$

Thus,

$$E(s) := \frac{1}{2}\left(e^{\frac{\alpha s^2}{2}}|f'|^2 + \frac{c^2}{4}|f|^2\right) = E_0, \qquad \forall\, s \in \mathbb{R}, \tag{4.27}$$

with E_0 a constant defined by the value of $E(0)$. Indeed, the energies associated to the initial conditions (4.24)–(4.26) are respectively

$$E_{0,1} = \frac{c^2}{8}, \qquad E_{0,2} = \frac{c^2}{4} \qquad \text{and} \qquad E_{0,3} = \frac{c^2}{4}. \tag{4.28}$$

[1]We write f instead $f_{c,\alpha}$ for notational simplicity.

It follows from (4.28) and the formulas in Lemma 4.9, that

$$\mathfrak{m}_{1,c,\alpha}(s) = 2|f_1(s)|^2 - 1, \quad \mathfrak{n}_{1,c,\alpha}(s) + i\mathfrak{b}_{1,c,\alpha}(s) = \frac{4}{c}e^{\alpha s^2/4}\bar{f}_1(s)f_1'(s), \tag{4.29}$$

$$\mathfrak{m}_{j,c,\alpha}(s) = |f_j(s)|^2 - 1, \quad \mathfrak{n}_{j,c,\alpha}(s) + i\mathfrak{b}_{j,c,\alpha}(s) = \frac{2}{c}e^{\alpha s^2/4}\bar{f}_j(s)f_j'(s), \quad j \in \{2,3\}. \tag{4.30}$$

Let us remark that by Lemma 4.9, formulas (4.29) and (4.30) are valid as long as $m_j > -1$, which is equivalent to the condition $|f_j| \neq 0$. However, the trihedron $\{\mathfrak{m}_{c,\alpha}, \mathfrak{n}_{c,\alpha}, \mathfrak{b}_{c,\alpha}\}$ is defined globally and f_j can also be extended globally as the solution of the linear equation (4.23). Then, it is simple to verify that the functions given by the l.h.s. of formulae (4.29) and (4.30) satisfy the Serret–Frenet system and hence, by the uniqueness of the solution, formulas (4.29) and (4.30) are valid for all $s \in \mathbb{R}$.

Unlike in the critical cases $\alpha = 0$ and $\alpha = 1$, if $\alpha \in (0,1)$, no explicit solutions are known for (4.23), and the term containing the exponential in this equation makes it difficult to use the Fourier analysis methods in [GRV03] to study analytically the behavior of the solutions. As fundamental step in the analysis of the behavior of the solutions to (4.23), we generalize an idea from [GV04] consisting in introducing new *real valued* variables z, h and y defined by

$$z = |f|^2, \qquad y = \mathrm{Re}(\bar{f}f') \qquad \text{and} \qquad h = \mathrm{Im}(\bar{f}f') \tag{4.31}$$

in terms of solutions f of (4.23), that satisfies the linear system

$$z' = 2y, \tag{4.32}$$

$$y' = \beta\frac{s}{2}h - \alpha\frac{s}{2}y + e^{-\alpha s^2/2}\left(2E_0 - \frac{c^2}{2}z\right), \tag{4.33}$$

$$h' = -\beta\frac{s}{2}y - \alpha\frac{s}{2}h. \tag{4.34}$$

In particular, it follows from (4.27) that for all $s \in \mathbb{R}$,

$$|f(s)| \le \frac{\sqrt{8E_0}}{c}, \; |f'(s)| \le \sqrt{2E_0}\, e^{-\alpha s^2/4}, \; |z(s)| \le \frac{8E_0}{c^2}, \; |h(s)| + |y(s)| \le \frac{8E_0}{c}e^{-\alpha s^2/4}. \tag{4.35}$$

As we will see later on, these variables are the "natural" ones in our problem, in the sense that the components of the tangent, normal and binormal vectors can be written in terms of these quantities as follows

$$\begin{cases} \mathfrak{m}_{1,c,\alpha} = 2z_1 - 1, & \mathfrak{n}_{1,c,\alpha} = \frac{4}{c}e^{\alpha s^2/4}\, y_1, \quad \mathfrak{b}_{1,c,\alpha} = \frac{4}{c}e^{\alpha s^2/4}\, h_1, \\ \mathfrak{m}_{j,c,\alpha} = z_j - 1, & \mathfrak{n}_{j,c,\alpha} = \frac{2}{c}e^{\alpha s^2/4}\, y_j, \quad \mathfrak{b}_{j,c,\alpha} = \frac{2}{c}e^{\alpha s^2/4}\, h_j, \quad j \in \{2,3\}, \end{cases}$$

where we have used the evident notation $z_j = |f_j|^2$, $y_j = \mathrm{Re}(\bar{f}_j f_j')$ and $h_j = \mathrm{Im}(\bar{f}_j f_j')$.

It is important to emphasize that, in order to obtain error bounds in the asymptotic analysis independent of the damping parameter α, we need uniform estimates for $\alpha \in [0,1]$. We begin our analysis by establishing the following result.

Proposition 4.10. *Let $c > 0$ and $\alpha \in [0,1]$. Let (z, y, h) be a solution of the system (4.32)–(4.34). Then the limit $z_\infty := \lim_{s\to\infty} z(s)$ exists and for all $s \geq s_0 := 4\sqrt{8+c^2}$, we have*

$$z(s) - z_\infty = -\frac{4}{s}(\alpha y + \beta h) - \frac{4\gamma}{s^2} e^{-\alpha s^2/2} + R_0(s), \tag{4.36}$$

where

$$|R_0(s)| \leq C(E_0, c)\frac{e^{-\alpha s^2/4}}{s^3}, \tag{4.37}$$

and $\gamma := 2E_0 - c^2 z_\infty/2$.

The fact that z admits a limit at infinity, can be seen plugging (4.32) into (4.34), and integrating from 0 to some $s > 0$, so that

$$z(s) - \frac{1}{s}\int_0^s z(\sigma)\,d\sigma = -\frac{4}{\beta s}\left(h(s) - h(0) + \frac{\alpha}{2}\int_0^s \sigma h(\sigma)\,d\sigma\right). \tag{4.38}$$

Also, notice that

$$\frac{d}{ds}\left(\frac{1}{s}\int_0^s z(\sigma)\,d\sigma\right) = -\frac{4}{\beta s^2}\left(h(s) - h(0) + \frac{\alpha}{2}\int_0^s \sigma h(\sigma)\,d\sigma\right). \tag{4.39}$$

Now, since from (4.35), we have $|h(s)| \leq (8E_0/c)\,e^{-\alpha s^2/4}$, both h and $\alpha\int_0^s \sigma h(\sigma)\,d\sigma$ are bounded functions, thus from (4.39) it follows that the limit of $\frac{1}{s}\int_0^s z$ exists, as $s \to \infty$. Hence, (4.38) and previous observations imply that the limit $z_\infty := \lim_{s\to\infty} z(s)$ exists and furthermore

$$z_\infty := \lim_{s\to\infty} z(s) = \lim_{s\to\infty}\frac{1}{s}\int_0^s z(\sigma).$$

The asymptotics in (4.36)–(4.37) follow integrating (4.39), using (4.32)–(4.34) and several integrations by parts.

Formula (4.36) in Proposition 4.10 gives z in terms of y and h. Therefore, we can reduce our analysis to that of the variables y and h. In fact, a first attempt could be to define $w = y + ih$, so that from (4.33) and (4.34), we see that w solves

$$\left(w e^{(\alpha+i\beta)s^2/4}\right)' = e^{(-\alpha+i\beta)s^2/4}\left(\gamma - \frac{c^2}{2}(z - z_\infty)\right).$$

From (4.35) and (4.2.3), we see that the limit $w_* = \lim_{s\to\infty} w(s)e^{(\alpha+i\beta)s^2/4}$ exists (at least when $\alpha \neq 0$), and integrating (4.2.3) from some $s > 0$ to ∞, we find that

$$w(s) = e^{-(\alpha+i\beta)s^2/4}\left(w_* - \int_s^\infty e^{(-\alpha+i\beta)\sigma^2/4}\left(\gamma - \frac{c^2}{2}(z - z_\infty)\right)d\sigma\right).$$

In order to obtain an asymptotic expansion, we need to estimate $\int_s^\infty e^{(-\alpha+i\beta)\sigma^2/4}(z - z_\infty)$, for s large. This can be achieved using (4.35),

$$\left|\int_s^\infty e^{(-\alpha+i\beta)\sigma^2/4}(z - z_\infty)\,d\sigma\right| \leq C(E_0, c)\int_s^\infty \frac{e^{-\alpha\sigma^2/2}}{\sigma}\,d\sigma \tag{4.40}$$

and the asymptotic expansion

$$\int_s^\infty \frac{e^{-\alpha\sigma^2/2}}{\sigma}\,d\sigma = e^{-\alpha s^2/2}\left(\frac{1}{\alpha s^2} - \frac{2}{\alpha^2 s^4} + \frac{8}{\alpha^3 s^6} + \cdots\right).$$

However, this estimate diverges as $\alpha \to 0$. The problem is that the bound used in obtaining (4.40) does not take into account the cancellations due to the oscillations. Therefore, and in order to obtain the asymptotic behavior of z, y and h valid for all $\alpha \in [0,1]$, we need a more refined analysis. The next key proposition provides the asymptotics for the system (4.32)–(4.34), considering the cancellations due to oscillations (see Lemma 4.12 below). Theorems 4.3 and 4.4 are consequences of these asymptotics.

Proposition 4.11. *There is s_1 such that for all $s \geq s_1$,*

$$y(s) = be^{-\alpha s^2/4}\sin(\phi(s_1;s)) - \frac{2\alpha\gamma}{s}e^{-\alpha s^2/2} + O\left(\frac{e^{-\alpha s^2/2}}{\beta^2 s^2}\right),$$
$$h(s) = be^{-\alpha s^2/4}\cos(\phi(s_1;s)) - \frac{2\beta\gamma}{s}e^{-\alpha s^2/2} + O\left(\frac{e^{-\alpha s^2/2}}{\beta^2 s^2}\right),$$

where

$$\phi(s_1;s) = a + \beta\int_{s_1^2/4}^{s^2/4}\sqrt{1 + c^2\frac{e^{-2\alpha t}}{t}}\,dt,$$

$a \in [0,2\pi)$ is a real constant, and b is given by $b^2 = (2E_0 - c^2 z_\infty/4)z_\infty$.

Let us give some details about the proof of this proposition. First, notice that plugging the expression for $z(s) - z_\infty$ in (4.36) into (4.33), and introducing the new variables, $u(t) = e^{\alpha t}y(2\sqrt{t})$ and $v(t) = e^{\alpha t}h(2\sqrt{t})$, we recast the system as

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} \alpha K & \beta(1+K) \\ -\beta & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{4.41}$$

with

$$K = \frac{c^2 e^{-2\alpha t}}{t}, \quad F = \gamma\frac{e^{-\alpha t}}{\sqrt{t}} + \frac{e^{-\alpha t}}{\sqrt{t}}R_1(2\sqrt{t}), \quad R_1(s) = -\frac{c^2}{2}e^{-\alpha s^2/2}R_0(s) + \frac{2c^2\gamma e^{-\alpha s^2}}{s^2}$$

In this way, we can see (4.41) as a non-autonomous system. It is straightforward to check that the matrix

$$A = \begin{pmatrix} \alpha K & \beta(1+K) \\ -\beta & 0 \end{pmatrix}$$

is diagonalizable, i.e. $A = PDP^{-1}$, with $D = \operatorname{diag}(\lambda_+, \lambda_-)$

$$P = \begin{pmatrix} -\frac{\alpha K}{2\beta} - i\Delta^{1/2} & -\frac{\alpha K}{2\beta} + i\Delta^{1/2} \\ 1 & 1 \end{pmatrix}, \quad \lambda_\pm = \frac{\alpha K}{2} \pm i\beta\Delta^{1/2}, \quad \Delta = 1 + K - \frac{\alpha^2 K^2}{4\beta^2}.$$

Thus, defining $\omega = (\omega_1, \omega_2) = P^{-1}(u, v)$, we see that $\omega_1 = \bar{\omega}_2$, and we get

$$\omega_1(t) = e^{\int_{t_1}^t \lambda_+} \left(\omega_1(t_1) + \omega_\infty - \int_t^\infty e^{-\int_{t_1}^\tau \lambda_+} G(\tau)\, d\tau \right),$$

for some $\omega_\infty \in \mathbb{C}$, where

$$G(t) = i \frac{\alpha K'}{4\beta \Delta^{1/2}} (\omega_1 + \bar{\omega}_1) - \frac{\Delta'}{4\Delta} (\omega_1 - \bar{\omega}_1) + i \frac{F}{2\Delta^{1/2}}.$$

Using Lemma 4.12, a careful analyze allows to deduce that

$$\omega_1(t) = \frac{b}{2} e^{i(\beta I(t)+a)} - \frac{\gamma(\beta + i\alpha)e^{-\alpha t}}{2t^{1/2}} + R_{\omega_1}(t) \quad \text{with} \quad |R_{\omega_1}(t)| \le \frac{C(E_0, c)e^{-\alpha t}}{\beta^2 t},$$

for some real constants a and b, with $b \ge 0$ and $a \in [0, 2\pi)$. Then the conclusion follows going back to the definition of ω, so that $(u, v) = P(\omega_1, \bar{\omega}_1)$.

In the proof of Proposition 4.11, we have used the following key lemma that establishes the control of certain integrals by exploiting their oscillatory character. The proof is based on repeated integration by parts, in the spirit of the method of stationary phase.

Lemma 4.12. *With the same notation as in the proof of Proposition 4.11.*

(i) *Let $g \in C^1((t_1, \infty))$ such that*

$$|g(t)| \le L/t^a \quad \text{and} \quad |g'(t)| \le L\left(\frac{\alpha}{t^a} + \frac{1}{t^{a+1}}\right),$$

for some constants L, $a > 0$. Then, for all $t \ge t_1$ and $l \ge 1$

$$\int_t^\infty e^{-\int_{t_1}^\tau \lambda_+} e^{-l\alpha\tau} g(\tau)\, d\tau = \frac{1}{(l\alpha + i\beta)} e^{-\int_{t_1}^t \lambda_+} e^{-l\alpha t} g(t) + G(t),$$

with $|G(t)| \le C(l, a, c) L e^{-l\alpha t}/(\beta t^a)$.

(ii) *If in addition $g \in C^2((t_1, \infty))$,*

$$|g'(t)| \le L/t^{a+1} \quad \text{and} \quad |g''(t)| \le L\left(\frac{\alpha}{t^{a+1}} + \frac{1}{t^{a+2}}\right),$$

then $|G(t)| \le C(l, a, c) L e^{-l\alpha t}/(\beta t^{a+1})$.

Here $C(l, a, c)$ is a positive constant depending only on l, a and c.

4.2.4 Elements of the proofs of Theorems 4.5 and 4.6

In this subsection, we provide some details about the dependence on the parameters c ad α of the limit vector $\boldsymbol{A}^+_{c,\alpha}$. In view of (4.29), (4.30) and (4.31), we reduce our study to the properties of

$$z_{\infty,c,\alpha} := \lim_{s\to\infty} |f_{c,\alpha}(s)|^2,$$

where $f_{c,\alpha}$ is a solution of (4.23) with $E_0 > 0$ defined in (4.27). We assume that the initial conditions $f_{c,\alpha}(0)$ and $f'_{c,\alpha}(0)$ depend smoothly on c and that they are independent of α, as in (4.24), (4.25) and (4.26). Hence, Theorem 4.5 is a consequence of the following result.

Proposition 4.13. *Let $\alpha \in [0,1]$. Then $z_{\infty,c,\alpha}$ is a continuous function of $c \in (0,\infty)$. Moreover, if $\alpha \in (0,1]$, the following estimate holds*

$$\left| z_{\infty,c,\alpha} - \left| f_{c,\alpha}(0) + \frac{f'_{c,\alpha}(0)\sqrt{\pi}}{\sqrt{\alpha+i\beta}} \right|^2 \right| \leq \frac{\sqrt{2E_0}c\pi}{\alpha} \left| f_{c,\alpha}(0) + \frac{f'_{c,\alpha}(0)\sqrt{\pi}}{\sqrt{\alpha+i\beta}} \right| + \left(\frac{\sqrt{2E_0}c\pi}{2\alpha} \right)^2.$$

The idea of the proof of Proposition 4.13 is that multiplying (4.23) by $e^{(\alpha+i\beta)s^2/4}$, we get

$$(f' e^{(\alpha+i\beta)s^2/4})' = -\frac{c^2}{4} f(s) e^{(-\alpha+i\beta)s^2/4}.$$

Hence, integrating twice, we have $f(s) = f(0) + G(s) + F(s)$, where

$$G(s) = f'(0) \int_0^s e^{-(\alpha+i\beta)\sigma^2/4}\, d\sigma \quad \text{and} \quad F(s) = -\frac{c^2}{4} \int_0^s e^{-(\alpha+i\beta)\sigma^2/4} \int_0^\sigma e^{(-\alpha+i\beta)\tau^2/4} f(\tau)\, d\tau\, d\sigma.$$

The result follows using the bounds in (4.35), letting $s \to \infty$ and noticing that

$$\lim_{s\to\infty} G(s) = f'(0) \int_0^\infty e^{-(\alpha+i\beta)\sigma^2/4}\, d\sigma = f'(0) \frac{\sqrt{\pi}}{\sqrt{\alpha+i\beta}}.$$

The dependence of $z_{\infty,c,\alpha}$ on α, for fixed c, is more involved. The first step is to get some bounds on the derivative with respect to α of the function $f_{c,\alpha}$.

Lemma 4.14. *Let $\alpha \in (0,1)$. There exists a constant $C(c)$, depending on c but not on α, such that for all $s \geq 0$,*

$$\left| \frac{\partial}{\partial\alpha} f_{c,\alpha}(s) \right| \leq \frac{C(c)s^2}{\sqrt{\alpha(1-\alpha)}}, \qquad \left| \frac{\partial^2}{\partial\alpha\partial s} f_{c,\alpha}(s) \right| \leq e^{-\alpha s^2/4} \frac{C(c)s^2}{\sqrt{\alpha(1-\alpha)}}. \tag{4.42}$$

To obtain this estimate, it is enough to differentiate (4.23) with respect to α, multiply by $\partial_s\bar{\eta}$, take real part and integrate, to get

$$\frac{1}{2}\partial_s\left(|\partial_s\eta|^2\right) + \frac{\alpha s}{2}|\partial_s\eta|^2 + \frac{c^2}{8}\partial_s\left(|\eta|^2\right) e^{-\alpha s^2/2} = \mathrm{Re}(g\partial_s\bar{\eta}).$$

Then (4.42) follows by invoking a Gronwall argument.

To conclude the idea of the proof of Theorem 4.6, let us set $z(s,\alpha) = z_{c,\alpha}(s)$, $y(s,\alpha) = y_{c,\alpha}(s)$, $z_\infty(\alpha) = z_{\infty,c,\alpha}$. We also fix $\alpha_1, \alpha_2 \in (0,1]$ with $\alpha_1 < \alpha_2$. By classical results from the ODE theory, the functions $y(s,\alpha)$, and $z(s,\alpha)$ are smooth in $\mathbb{R}\times[0,1)$ and continuous in $\mathbb{R}\times[0,1]$ (see e.g. [CL55, Har64]). Hence, integrating (4.32) with respect to s, we deduce that

$$z_\infty(\alpha_2) - z_\infty(\alpha_1) = 2\int_0^\infty (y(s,\alpha_2) - y(s,\alpha_1))\, ds = 2\int_0^\infty \int_{\alpha_1}^{\alpha_2} \partial_\mu y(s,\mu)\, d\mu\, ds.$$

To estimate the last integral, we remark that differentiating the relation $y(s,\mu) = \mathrm{Re}(\bar{f}(s,\mu)\partial_s f(s,\mu))$, with respect to μ, and using (4.35) and (4.42), we have

$$|\partial_\mu y(s,\mu)| \leq |\partial_\mu f(s,\mu)||\partial_s f(s,\mu)| + |f(s,\mu)||\partial^2_{\mu s} f(s,\mu)| \leq e^{-\mu s^2/4}\frac{C(c)s^2}{\sqrt{\mu(1-\mu)}}.$$

Then, using that

$$\int_{\alpha_1}^{\alpha_2} |\partial_\mu y(s,\mu)|\, d\mu \leq C(c)\frac{s^2}{\sqrt{\alpha_1}} 2\left(\sqrt{1-\alpha_1}e^{-\alpha_1 s^2/4} - \sqrt{1-\alpha_2}e^{-\alpha_2 s^2/4}\right),$$

and analyzing the involved integrals, we can deduce the estimates in Theorem 4.6.

4.3 The Cauchy problem for LLG in BMO

A natural question in the study of the stability properties of the family of solutions $(\boldsymbol{m}_{c,\alpha})_{c>0}$ is whether or not it is possible to develop a well-posedness theory for the Cauchy problem for (4.1) in a functional framework that allows us to handle initial conditions of the type (4.11). In view of (4.11) and (4.12), such a framework should allow some "rough" functions (i.e. function spaces beyond the "classical" energy ones) and step functions.

A few remarks about previously known results in this setting are in order. In the case $\alpha > 0$, global well-posedness results for (4.1) have been established in $N \geq 2$ by Melcher [Mel12] and by Lin, Lai and Wang [LLW15] for initial conditions with a smallness condition on the gradient in the $L^N(\mathbb{R}^N)$ and the Morrey $M^{2,2}(\mathbb{R}^N)$ norm[2], respectively. Therefore, these results do not apply to the initial condition $\boldsymbol{m}^0_{c,\alpha}$. When $\alpha = 1$, global well-posedness results for the heat flow for harmonic maps (HFHM) have been obtained by Koch and Lamm [KL12] for an initial condition L^∞-close to a point and improved to an initial data with small BMO semi-norm by Wang [Wan11]. The ideas used in [KL12] and [Wan11] rely on techniques introduced by Koch and Tataru [KT01] for the Navier–Stokes equation. Since $\boldsymbol{m}^0_{c,\alpha}$ has a small BMO semi-norm if c is small, the results in [Wan11] apply to the case $\alpha = 1$.

In this section we explain the main results in [dLG19] that allow us to adapt and extend the techniques developed in [KL12, KT01, Wan11] to prove a global well-posedness result for (4.1) with $\alpha \in (0,1]$, for data $\boldsymbol{m}^0$ in $L^\infty(\mathbb{R}^N;\mathbb{S}^2)$ with small BMO semi-norm. As an application of these results, we can establish the stability of the family of self-similar solutions $(\boldsymbol{m}_{c,\alpha})_{c>0}$ and derive further properties for these solutions. In particular, we can prove the existence of multiple smooth solutions of (4.1) associated with the same initial condition, provided that α is close to one.

Our approach to study the Cauchy problem for (4.1) consists in analyzing the Cauchy problem for the associated dissipative quasilinear Schrödinger equation through the stereographic projection, and then "transferring" the results back to the original equation. To this

[2] We recall that $v \in M^{2,2}(\mathbb{R}^N)$ if $v \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $\|v\|_{M^{2,2}} := \sup_{\substack{x\in\mathbb{R}^N\\ r>0}} \frac{1}{r^{(N-2)/2}}\|v\|_{L^2(B_r(x))} < \infty$.

end, we introduce the stereographic projection from the South Pole:

$$\mathcal{P}(\boldsymbol{m}) = \frac{m_1 + im_2}{1+m_3}.$$

Let $\boldsymbol{m}$ be a smooth solution of (4.1) with $m_3 > -1$, then its stereographic projection $u = \mathcal{P}(\boldsymbol{m})$ satisfies the quasilinear dissipative Schrödinger equation (see e.g. [LN84] for details)

$$iu_t + (\beta - i\alpha)\Delta u = 2(\beta - i\alpha)\frac{\bar{u}(\nabla u)^2}{1+|u|^2}. \tag{DNLS}$$

At least formally, the Duhamel formula gives the integral equation:

$$u(x,t) = S_\alpha(t)u^0 + \int_0^t S_\alpha(t-s)g(u)(s)\,ds, \tag{IDNLS}$$

where $u^0 = u(\cdot, 0)$ corresponds to the initial condition,

$$g(u) = -2i(\beta - i\alpha)\frac{\bar{u}(\nabla u)^2}{1+|u|^2}$$

and $S_\alpha(t)$ is the dissipative Schrödinger semigroup (also called the complex Ginzburg–Landau semigroup) given by $S_\alpha(t)\phi = e^{(\alpha+i\beta)t\Delta}\phi$, i.e.

$$(S_\alpha(t)\phi)(x) = \int_{\mathbb{R}^N} G_\alpha(x-y,t)\phi(y)\,dy, \quad \text{with} \quad G_\alpha(x,t) = \frac{e^{-\frac{|x|^2}{4(\alpha+i\beta)t}}}{(4\pi(\alpha+i\beta)t)^{N/2}}. \tag{4.43}$$

One difficulty in studying (IDNLS) is to handle the term $g(u)$. We see that $|g(u)| \le |\nabla u|^2$, so we need to control $|\nabla u|^2$. Koch and Taratu dealt with a similar problem when studying the well-posedness for the Navier–Stokes equation in [KT01]. Their approach was to introduce some new spaces related to BMO and BMO^{-1}. Later, Koch and Lamm [KL12], and Wang [Wan11] have adapted these spaces to study some geometric flows. Following these ideas, we define the Banach spaces

$$\begin{aligned} &X(\mathbb{R}^N \times \mathbb{R}^+; F) = \{v : \mathbb{R}^N \times \mathbb{R}^+ \to F \ : \ v, \nabla v \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}^+),\ \|v\|_X < \infty\} \qquad \text{and} \\ &Y(\mathbb{R}^N \times \mathbb{R}^+; F) = \{v : \mathbb{R}^N \times \mathbb{R}^+ \to F \ : v \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}^+),\ \|v\|_Y < \infty\}, \end{aligned}$$

where

$$\begin{aligned} \|v\|_X &:= \sup_{t>0} \|v\|_{L^\infty} + [v]_X, \qquad \text{with} \\ [v]_X &:= \sup_{t>0} \sqrt{t}\|\nabla v\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^N \\ r>0}} \left(\frac{1}{r^N}\int_{Q_r(x)} |\nabla v(y,t)|^2\,dt\,dy\right)^{\frac{1}{2}}, \qquad \text{and} \\ \|v\|_Y &= \sup_{t>0} t\|v\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^N \\ r>0}} \frac{1}{r^N}\int_{Q_r(x)} |v(y,t)|\,dt\,dy. \end{aligned}$$

Here $Q_r(x)$ denotes the parabolic ball $Q_r(x) = B_r(x) \times [0, r^2]$ and F is either $\mathbb{C}$ or $\mathbb{R}^3$. The absolute value stands for the complex absolute value if $F = \mathbb{C}$ and for the euclidean norm if $F = \mathbb{R}^3$. We denote with the same symbol the absolute value in F and F^3. Here and in the sequel we will omit the domain in the norms and semi-norms when they are taken in the whole space, for example $\|\cdot\|_{L^p}$ stands for $\|\cdot\|_{L^p(\mathbb{R}^N)}$, for $p \in [1, \infty]$.

The spaces X and Y are related to the spaces $\mathrm{BMO}(\mathbb{R}^N)$ and $\mathrm{BMO}^{-1}(\mathbb{R}^N)$ and are well-adapted to study problems involving the heat semigroup $S_1(t) = e^{t\Delta}$. In order to establish the properties of the semigroup $S_\alpha(t)$ with $\alpha \in (0, 1]$, we introduce the spaces $\mathrm{BMO}_\alpha(\mathbb{R}^N)$ and $\mathrm{BMO}_\alpha^{-1}(\mathbb{R}^N)$ as the space of distributions $f \in S'(\mathbb{R}^N; F)$ such that the semi-norm and norm given respectively by

$$[f]_{BMO_\alpha} = \sup_{\substack{x\in\mathbb{R}^N \\ r>0}} \left(\frac{1}{r^N} \int_{Q_r(x)} |\nabla S_\alpha(t) f|^2 \right)^{\frac{1}{2}}, \quad \|f\|_{BMO_\alpha^{-1}} = \sup_{\substack{x\in\mathbb{R}^N \\ r>0}} \left(\frac{1}{r^N} \int_{Q_r(x)} |S_\alpha(t) f|^2 \right)^{\frac{1}{2}},$$

are finite.

On the one hand, the Carleson measure characterization of BMO functions (see [Ste93, Chapter 4] and [LR02, Chapter 10]) yields that for fixed $\alpha \in (0, 1]$, $\mathrm{BMO}_\alpha(\mathbb{R}^N)$ coincides with the classical $\mathrm{BMO}(\mathbb{R}^N)$ space[3], that is for all $\alpha \in (0, 1]$ there exists a constant $\Lambda > 0$ depending only on α and N such that

$$\Lambda [f]_{BMO} \le [f]_{BMO_\alpha} \le \Lambda^{-1} [f]_{BMO}.$$

On the other hand, Koch and Tataru proved in [KT01] that BMO^{-1} (or equivalently BMO_1^{-1}, using our notation) can be characterized as the space of derivatives of functions in BMO. A straightforward generalization of their argument shows that the same result holds for BMO_α^{-1}. Hence, using the Carleson measure characterization theorem, we conclude that BMO_α^{-1} coincides with the space BMO^{-1} and that there exists a constant $\tilde{\Lambda} > 0$, depending only on α and N, such that

$$\tilde{\Lambda} \|f\|_{BMO^{-1}} \le \|f\|_{BMO_\alpha^{-1}} \le \tilde{\Lambda}^{-1} \|f\|_{BMO^{-1}}.$$

The above remarks allows us to use several of the estimates proved in [KL12, KT01, Wan11] in the case $\alpha = 1$, to study the integral equation (IDNLS) by using a fixed-point approach.

The next result provides the global well-posedness of the Cauchy problem for (IDNLS) with small initial data in $\mathrm{BMO}(\mathbb{R}^N)$.

---

[3]

$$BMO(\mathbb{R}^N) = \{ f : \mathbb{R}^N \times [0, \infty) \to F \ : f \in L^1_{\mathrm{loc}}(\mathbb{R}^N),\ [f]_{BMO} < \infty \},$$

with the semi-norm

$$[f]_{BMO} = \sup_{\substack{x\in\mathbb{R}^N \\ r>0}} \fint_{B_r(x)} |f(y) - f_{x,r}|\, dy, \quad \text{where } f_{x,r} = \fint_{B_r(x)} f(y)\, dy = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)\, dy.$$

Theorem 4.15 ([dLG19]). *Let $\alpha \in (0,1]$. There exist constants $C, K \geq 1$ such that for every $L \geq 0$, $\varepsilon > 0$, and $\rho > 0$ satisfying*

$$8C(\rho+\varepsilon)^2 \leq \rho, \tag{4.44}$$

if $u^0 \in L^\infty(\mathbb{R}^N;\mathbb{C})$, with

$$\|u^0\|_{L^\infty} \leq L \quad \text{and} \quad [u^0]_{BMO} \leq \varepsilon, \tag{4.45}$$

then there exists a unique solution $u \in X(\mathbb{R}^N \times \mathbb{R}^+;\mathbb{C})$ to (IDNLS) *such that*

$$[u]_X \leq K(\rho+\varepsilon). \tag{4.46}$$

Moreover,

(i) $\sup_{t>0} \|u\|_{L^\infty} \leq K(\rho+L)$.

(ii) $u \in C^\infty(\mathbb{R}^N \times \mathbb{R}^+)$ *and* (DNLS) *holds pointwise.*

(iii) $\lim\limits_{t\to 0^+} u(\cdot,t) = u^0$ *as tempered distributions.*

(iv) *(Dependence on the initial data) Assume that u and v are respectively solutions to* (IDNLS) *fulfilling* (4.46) *with initial conditions u^0 and v^0 satisfying* (4.45)*. Then*

$$\|u-v\|_X \leq 6K\|u^0-v^0\|_{L^\infty}.$$

Although condition (4.44) appears naturally from the fixed-point used in the proof, it may be no so clear at first glance. To better understand it, let us define for $C > 0$

$$\mathcal{S}(C) = \{(\rho,\varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^+ : C(\rho+\varepsilon)^2 \leq \rho\}.$$

We see that if $(\rho,\varepsilon) \in \mathcal{S}(C)$, then $\rho, \varepsilon > 0$ and

$$\varepsilon \leq \frac{\sqrt{\rho}}{\sqrt{C}} - \rho. \tag{4.47}$$

Therefore, the set $\mathcal{S}(C)$ is non-empty and bounded. The shape of this set is depicted in Figure 4.2. In particular, we infer from (4.47) that if $(\rho,\varepsilon) \in \mathcal{S}(C)$, then $\rho \leq \frac{1}{C}$ and $\varepsilon \leq \frac{1}{4C}$. In addition, if $\tilde{C} \geq C$, then

$$\mathcal{S}(\tilde{C}) \subseteq \mathcal{S}(C).$$

Moreover, taking for instance $\rho = 1/(32C)$, Theorem 5.6 asserts that for fixed $\alpha \in (0,1]$, we can take for instance $\varepsilon = 1/(32C)$ (that depends on α and N, but not on the L^∞-norm of the initial data) such that for any given initial condition $u^0 \in L^\infty(\mathbb{R}^N)$ with $[u^0]_{BMO} \leq \varepsilon$, there exists a global (smooth) solution $u \in X(\mathbb{R}^N \times \mathbb{R}^+;\mathbb{C})$ of (DNLS). Notice that u^0 is allowed to have a large L^∞-norm as long as $[u^0]_{BMO}$ is sufficiently small; this is a weaker requirement that asking for the L^∞-norm of u^0 to be sufficiently small, since $[f]_{BMO} \leq 2\|f\|_{L^\infty}$, for all $f \in L^\infty(\mathbb{R}^N)$.

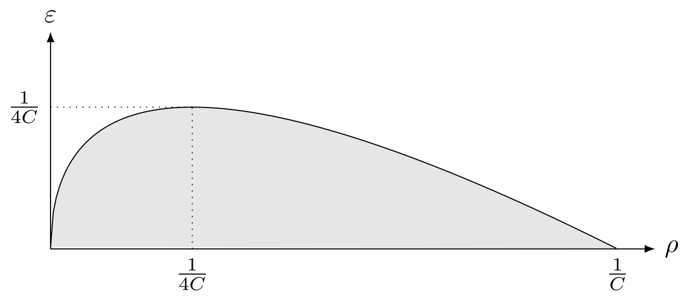


Figure 4.2: The shape of the set $\mathcal{S}(C)$.

Remark 4.16. The smallness condition in (4.46) is necessary for the uniqueness of the solution. As we will see in Theorem 4.24, at least in dimension one, it is possible to construct multiple solutions of (IDNLS) in $X(\mathbb{R}^N \times \mathbb{R}^+; \mathbb{C})$, if α is close enough to 1.

The proof of Theorem 5.6 uses fixed-point argument on the ball

$$\mathcal{B}_\rho(u^0) = \{u \in X(\mathbb{R}^N \times \mathbb{R}^+; \mathbb{C}) : \|u - S_\alpha(t)u^0\|_X \leq \rho\},$$

for some $\rho > 0$ depending on the size of the initial data. Indeed, we can write (IDNLS) as $u(t) = \mathcal{T}_{u^0}(u)(t)$, where

$$\mathcal{T}_{u^0}(u)(t) = S_\alpha(t)u^0 + T(g(u))(t) \quad \text{and} \quad T(f)(t) = \int_0^t S_\alpha(t-s)f(s)\,ds.$$

There are three steps to establish that $\mathcal{T}_{u^0}$ is a contraction on $\mathcal{B}_\rho(u^0)$. The first one is to prove that there exists $C_0 > 0$ such that for all $f \in BMO_\alpha^{-1}(\mathbb{R}^N)$,

$$\sup_{t>0} \sqrt{t}\|S_\alpha(t)f\|_{L^\infty(\mathbb{R}^N)} \leq C_0\|f\|_{BMO_\alpha^{-1}}.$$

The proof in the case $\alpha = 1$ is done in [LR02, Lemma 16.1]. For $\alpha \in (0,1)$, decomposing $S_\alpha(t) = S_\alpha(t-s)S_\alpha(s)$ and using the decay properties of the kernel G_α associated with the operators $S_\alpha(t)$ (see (4.43)), we can check that the same proof still applies. Generalizing some arguments given in [KT01] and [Wan11], we have that there exists $C_1 \geq 1$ such that for all $f \in Y(\mathbb{R}^N \times \mathbb{R}^+; \mathbb{C})$,

$$\|T(f)\|_X \leq C_1\|f\|_Y.$$

The third step is to show that for all $u \in X(\mathbb{R}^N \times \mathbb{R}^+; \mathbb{C})$, we have $\|g(u)\|_Y \leq [u]_X^2$, which is a consequence of the definitions of the norms on X and Y.

By using the inverse of the stereographic projection $\mathcal{P}^{-1} : \mathbb{C} \to \mathbb{S}^2 \setminus \{0,0,-1\}$, that is explicitly given by $\boldsymbol{m} = (m_1, m_2, m_3) = \mathcal{P}^{-1}(u)$, with

$$m_1 = \frac{2\operatorname{Re} u}{1+|u|^2}, \quad m_2 = \frac{2\operatorname{Im} u}{1+|u|^2}, \quad m_3 = \frac{1-|u|^2}{1+|u|^2},$$

we can deduce from Theorem 5.6, the following global well-posedness result for (4.1).

Theorem 4.17 ([dLG19]). *Let $\alpha \in (0,1]$. There exist constants $C \geq 1$ and $K \geq 4$, such that for any $\delta \in (0,2]$, $\varepsilon_0 > 0$ and $\rho > 0$ such that*

$$8K^4 C\delta^{-4}(\rho + 8\delta^{-2}\varepsilon_0)^2 \leq \rho, \tag{4.48}$$

if $\boldsymbol{m}^0 = (m_1^0, m_2^0, m_3^0) \in L^\infty(\mathbb{R}^N; \mathbb{S}^2)$ satisfies

$$\inf_{\mathbb{R}^N} m_3^0 \geq -1 + \delta \quad \text{and} \quad [\boldsymbol{m}^0]_{BMO} \leq \varepsilon_0, \tag{4.49}$$

then there exists a unique solution $\boldsymbol{m} = (m_1, m_2, m_3) \in X(\mathbb{R}^N \times \mathbb{R}^+; \mathbb{S}^2)$ of (4.1) *such that*

$$\inf_{\substack{x\in\mathbb{R}^N\\ t>0}} m_3(x,t) \geq -1 + \frac{2}{1 + K^2(\rho + \delta^{-1})^2} \quad \text{and} \quad [\boldsymbol{m}]_X \leq 4K(\rho + 8\delta^{-2}\varepsilon_0). \tag{4.50}$$

Moreover, we have the following properties.

i) $\boldsymbol{m} \in C^\infty(\mathbb{R}^N \times \mathbb{R}^+; \mathbb{S}^2)$.

ii) $|\boldsymbol{m}(\cdot,t) - \boldsymbol{m}^0| \longrightarrow 0$ *in* $S'(\mathbb{R}^N)$ *as* $t \longrightarrow 0^+$.

iii) Assume that $\boldsymbol{m}$ and $\boldsymbol{n}$ are respectively smooth solutions to (IDNLS) *satisfying* (4.50) *with initial conditions $\boldsymbol{m}^0$ and $\boldsymbol{n}^0$ fulfilling* (4.49)*. Then*

$$\|\boldsymbol{m} - \boldsymbol{n}\|_X \leq 120K\delta^{-2}\|\boldsymbol{m}^0 - \boldsymbol{n}^0\|_{L^\infty}. \tag{4.51}$$

Remark 4.18. The restriction (4.48) on the parameters is similar to (4.44), but we need to include δ. To better understand the role of δ, we can proceed as before. Indeed, setting for $a, \delta > 0$,

$$\mathcal{S}_\delta(a) = \{(\rho, \varepsilon_0) \in \mathbb{R}^+ \times \mathbb{R}^+ : a\delta^{-4}(\rho + 8\delta^{-2}\varepsilon_0)^2 \leq \rho\},$$

we see that its shape is similar to the one in Figure 4.2. It is simple to verify that for any $(\rho, \varepsilon_0) \in \mathcal{S}_\delta(a)$, we have the bounds $\rho \leq \delta^4/a$ and $\varepsilon_0 \leq \delta^6/(32a)$, and the maximum value $\varepsilon_0^* = \delta^6/(32a)$ is attained at $\rho^* = \delta^4/(4a)$. Also, the sets are well-ordered, i.e. if $\tilde{a} \geq a > 0$, then $\mathcal{S}_\delta(\tilde{a}) \subseteq \mathcal{S}_\delta(a)$.

We emphasize that the first condition in (4.49) is rather technical. Indeed, we need the essential range of $\boldsymbol{m}^0$ to be far from the South Pole in order to use the stereographic projection. In the case $\alpha = 1$, Wang [Wan11] proved the global well-posedness using only the second restriction in (4.49). It is an open problem to determinate if this condition is necessary in the case $\alpha \in (0,1)$.

The choice of the South Pole is of course arbitrary. By using the invariance of (4.1) under rotations, we have the existence of solutions provided that the essential range of the initial condition $\boldsymbol{m}^0$ is far from an arbitrary point $\boldsymbol{Q} \in \mathbb{S}^2$. Precisely,

Corollary 4.19. *Let $\alpha \in (0,1]$, $\boldsymbol{Q} \in \mathbb{S}^2$, $\delta \in (0,2]$, and $\varepsilon_0, \rho > 0$ such that* (4.48) *holds. Given $\boldsymbol{m}^0 = (m_1^0, m_2^0, m_3^0) \in L^\infty(\mathbb{R}^N; \mathbb{S}^2)$ satisfying*

$$\inf_{\mathbb{R}^N} |\boldsymbol{m}^0 - \boldsymbol{Q}|^2 \geq 2\delta \quad \text{and} \quad [\boldsymbol{m}^0]_{BMO} \leq \varepsilon_0,$$

there exists a unique smooth solution $\boldsymbol{m} \in X(\mathbb{R}^N \times \mathbb{R}^+; \mathbb{S}^2)$ *of* (4.1) *with initial condition* $\boldsymbol{m}^0$ *such that*

$$\inf_{\substack{x\in\mathbb{R}^N\\ t>0}} |\boldsymbol{m}(x,t) - \boldsymbol{Q}|^2 \geq \frac{4}{1+K^2(\rho+\delta^{-1})^2} \quad \text{and} \quad [\boldsymbol{m}]_X \leq 4K(\rho + 8\delta^{-2}\varepsilon_0).$$

Notice that Theorem 4.17 provides the existence of a unique solution in the set defined by the conditions (4.50), and hence it does not exclude the possibility of the existence of other solutions not satisfying these conditions. In fact, as we will see in Theorem 4.24, one can prove a nonuniqueness result, at least in the case when α is close to 1.

We point out that our results are valid only for $\alpha > 0$. If we let $\alpha \to 0^+$, then our estimates blows up. Indeed, we use that the kernel associated with the Ginzburg–Landau semigroup $e^{(\alpha+i\beta)t\Delta}$ belongs to L^1 and its exponential decay. Therefore our techniques cannot be generalized (at least not in a simple way) to cover the critical case $\alpha = 0$. In particular, we cannot recover the stability results for the self-similar solutions in the case of Schrödinger maps proved by Banica and Vega in [BV09, BV12, BV13].

As mentioned before, in [LLW15] and [Mel12] some global well-posedness results for (4.1) with $\alpha \in (0,1]$ were proved for initial conditions with small gradient in $L^N(\mathbb{R}^N)$ and $M^{2,2}(\mathbb{R}^N)$, respectively. In view of the embeddings

$$L^N(\mathbb{R}^N) \subset M^{2,2}(\mathbb{R}^N) \subset BMO^{-1}(\mathbb{R}^N),$$

for $N \geq 2$, Theorem 4.17 can be seen as generalization of these results since it covers the case of less regular initial conditions. The arguments in [LLW15, Mel12] are based on the method of moving frames that produces a covariant complex Ginzburg–Landau equation.

Our existence and uniqueness result given by Theorem 4.17 requires the initial condition to be small in the BMO semi-norm. Without this condition, the solution could develop a singularity in finite time. In fact, in dimensions $N = 3,4$, Ding and Wang [DW07] have proved that for some smooth initial conditions with small (Dirichlet) energy, the associated solutions of (4.1) blow up in finite time.

Another consequence of Theorem 4.17 is the existence of self-similar solutions of expander type in $\mathbb{R}^N$, i.e. a solution $\boldsymbol{m}$ satisfying

$$\boldsymbol{m}(x,t) = \boldsymbol{m}(\lambda x, \lambda^2 t), \quad \forall \lambda > 0, \tag{4.52}$$

or, equivalently, $\boldsymbol{m}(x,t) = \boldsymbol{f}\Big(\frac{x}{\sqrt{t}}\Big)$, for some $\boldsymbol{f} : \mathbb{R}^N \longrightarrow \mathbb{S}^2$ profile of $\boldsymbol{m}$. In particular we have the relation $\boldsymbol{f}(y) = \boldsymbol{m}(y,1)$, for all $y \in \mathbb{R}^N$. From (4.52) we see that, at least formally, a necessary condition for the existence of a self-similar solution is that initial condition $\boldsymbol{m}^0$ be homogeneous of degree 0, i.e. $\boldsymbol{m}^0(\lambda x) = \boldsymbol{m}^0(x)$, for all $\lambda > 0$. Since the norm in $X(\mathbb{R}^N \times \mathbb{R}^+; \mathbb{R}^3)$ is invariant under this scaling, i.e. $\|\boldsymbol{m}_\lambda\|_X = \|\boldsymbol{m}\|_X, \quad \forall\lambda > 0$, where $\boldsymbol{m}_\lambda$ is defined by (4.52), Theorem 4.17 yields the following result concerning the existence of self-similar solutions.

Corollary 4.20. *With the same notations and hypotheses as in Theorem 4.17, assume also that $\boldsymbol{m}^0$ is homogeneous of degree zero. Then the solution $\boldsymbol{m}$ of* (4.1) *provided by Theorem 4.17 is self-similar. In particular there exists a smooth profile $\boldsymbol{f} : \mathbb{R}^N \to \mathbb{S}^2$ such that $\boldsymbol{m}(x,t) = \boldsymbol{f}(x/\sqrt{t})$, for all $x \in \mathbb{R}^N$ and $t > 0$.*

Remark 4.21. Analogously, Theorem 5.6 leads to the existence of self-similar solutions for (DNLS), provided that u^0 is a homogeneous function of degree zero.

For instance, in dimensions $N \geq 2$, Corollary 4.20 applies to the initial condition $\boldsymbol{m}^0(x) = \boldsymbol{H}(x/|x|)$, with $\boldsymbol{H}$ a Lipschitz map from $\mathbb{S}^{N-1}$ to $\mathbb{S}^2 \cap \{(x_1, x_2, x_3) : x_3 \geq -1/2\}$, provided that the Lipschitz constant is small enough, since $[\boldsymbol{m}^0]_{BMO} \leq 4\|\boldsymbol{H}\|_{\mathrm{Lip}}$.

Other authors have considered self-similar solutions for the harmonic map flow (i.e. (4.1) with $\alpha = 1$) in different settings. Actually, equation (HFHM) can be generalized for maps $\boldsymbol{m} : \mathcal{M} \times \mathbb{R}^+ \to \mathcal{N}$, with $\mathcal{M}$ and $\mathcal{N}$ Riemannian manifolds. Biernat and Bizoń [BB11] established results when $\mathcal{M} = \mathcal{N} = \mathbb{S}^d$ and $3 \leq d \leq 6$. Also, Germain and Rupflin [GR11] have investigated the case $\mathcal{M} = \mathbb{R}^d$ and $\mathcal{N} = \mathbb{S}^d$, in $d \geq 3$. In both works the analysis is done only for equivariant solutions and does not cover the case $\mathcal{M} = \mathbb{R}^N$ and $\mathcal{N} = \mathbb{S}^2$.

4.4 LLG with a jump initial data

In this section we want to apply our well-posedness result to the self-similar solutions $\boldsymbol{m}_{c,\alpha}$ with initial conditions $\boldsymbol{m}^0_{c,\alpha} := \boldsymbol{A}^+_{c,\alpha}\chi_{\mathbb{R}^+} + \boldsymbol{A}^-_{c,\alpha}\chi_{\mathbb{R}^-}$. Let us remark that the first term in the definition of $[\boldsymbol{v}]_X$ allows us to capture a blow-up rate of $1/\sqrt{t}$ for $\|\nabla v(t)\|_{L^\infty}$, as $t \to 0^+$. This is exactly the blow-up rate for the self-similar solutions found in (4.10). The integral term in the semi-norm $[\cdot]_X$ is also well-adapted to these solutions. Indeed, for any $\alpha \in (0,1]$ and $c \geq 0$, we have

$$[\boldsymbol{m}^0_{c,\alpha}]_{BMO} \leq 2c\sqrt{2\pi}/\sqrt{\alpha} \quad \text{and} \quad [\boldsymbol{m}_{c,\alpha}]_X \leq 4c/\alpha^{\frac{1}{4}}. \tag{4.53}$$

Let us start by considering a more general problem: LLG equation, in dimension one, with a jump initial data, defined as follows

$$\begin{cases} \partial_t \boldsymbol{m} = \beta \boldsymbol{m} \times \partial_{xx}\boldsymbol{m} - \alpha \boldsymbol{m} \times (\boldsymbol{m} \times \partial_{xx}\boldsymbol{m}), & \text{on} \quad \mathbb{R} \times \mathbb{R}^+, \\ \boldsymbol{m}^0_{\boldsymbol{A}^\pm} := \boldsymbol{A}^+\chi_{\mathbb{R}^+} + \boldsymbol{A}^-\chi_{\mathbb{R}^-}, \end{cases}$$

where $\boldsymbol{A}^\pm$ are two given unitary vectors in $\mathbb{S}^2$. In the context of the initial value problem (4.54), the smallness condition in the BMO semi-norm is equivalent to the smallness of the angle between $\boldsymbol{A}^+$ and $\boldsymbol{A}^-$. From Theorem 4.17 we can establish two important consequences of (4.54).

The first one is the that from the uniqueness statement in Theorem 4.17, we can deduce that the solution to (4.54) provided by Theorem 4.17 is a rotation of a self-similar solution $\boldsymbol{m}_{c,\alpha}$ for an appropriate value of c. Precisely,

Theorem 4.22 ([dLG19]). *Let $\alpha \in (0,1]$. There exist $L_1, L_2 > 0$, $\delta^* \in (-1,0)$ and $\vartheta^* > 0$ such that the following holds. Let $\boldsymbol{A}^+$, $\boldsymbol{A}^- \in \mathbb{S}^2$ and let ϑ be the angle between them. If $0 < \vartheta \leq \vartheta^*$, then there exists a solution $\boldsymbol{m}$ of* (4.1) *with initial condition $\boldsymbol{m}^0_{\boldsymbol{A}^\pm}$. Moreover, there exists $0 < c < \frac{\sqrt{\alpha}}{2\sqrt{\pi}}$, such that $\boldsymbol{m}$ coincides up to a rotation with the self-similar solution $\boldsymbol{m}_{c,\alpha}$, i.e. there exists $\mathcal{R} \in SO(3)$, depending only on $\boldsymbol{A}^+$, $\boldsymbol{A}^-$, α and c, such that $\boldsymbol{m} = \mathcal{R}\boldsymbol{m}_{c,\alpha}$, and $\boldsymbol{m}$ is the unique solution satisfying*

$$\inf_{\substack{x\in\mathbb{R}\\ t>0}} m_3(x,t) \geq \delta^* \quad \text{and} \quad [\boldsymbol{m}]_X \leq L_1 + L_2 c.$$

The second one concerns the stability of the self-similar solutions. Indeed, from the dependence of the solution with respect to the initial data established in (4.51) and the estimates in (4.53), we obtain the following result: For any given $\boldsymbol{m}^0 \in \mathbb{S}^2$ close enough to $\boldsymbol{m}^0_{\boldsymbol{A}^\pm}$, the solution $\boldsymbol{m}$ of (4.1) associated with $\boldsymbol{m}^0$ given by Theorem 4.17 must remain close to a rotation of a self-similar solution $\boldsymbol{m}_{c,\alpha}$, for some $c > 0$. In particular, $\boldsymbol{m}$ remains close to a self-similar solution. The precise statement is provided in the following theorem.

Theorem 4.23 ([dLG19]). *Let $\alpha \in (0,1]$. There exist constants $L_1, L_2, L_3 > 0$, $\delta^* \in (-1,0)$, $\vartheta^* > 0$ such that the following holds. Let $\boldsymbol{A}^+$, $\boldsymbol{A}^- \in \mathbb{S}^2$ with angle ϑ between them. If $0 < \vartheta \leq \vartheta^*$, then there is $c > 0$ such that for every $\boldsymbol{m}^0$ satisfying*

$$\|\boldsymbol{m}^0 - \boldsymbol{m}^0_{\boldsymbol{A}^\pm}\|_{L^\infty} \leq \frac{c\sqrt{\pi}}{2\sqrt{\alpha}},$$

there exists $\mathcal{R} \in SO(3)$, depending only on $\boldsymbol{A}^+$, $\boldsymbol{A}^-$, α and c, such that there is a unique global smooth solution $\boldsymbol{m}$ of (4.1) *with initial condition $\boldsymbol{m}^0$ that satisfies*

$$\inf_{\substack{x\in\mathbb{R}\\ t>0}} (\mathcal{R}\boldsymbol{m})_3(x,t) \geq \delta^* \quad \text{and} \quad [\boldsymbol{m}]_X \leq L_1 + L_2 c.$$

Moreover,

$$\|\boldsymbol{m} - \mathcal{R}\boldsymbol{m}_{c,\alpha}\|_X \leq L_3 \|\boldsymbol{m}^0 - \boldsymbol{m}^0_{\boldsymbol{A}^\pm}\|_{L^\infty}.$$

In particular,

$$\|\partial_x \boldsymbol{m} - \partial_x \mathcal{R}\boldsymbol{m}_{c,\alpha}\|_{L^\infty} \leq \frac{L_3}{\sqrt{t}} \|\boldsymbol{m}^0 - \boldsymbol{m}^0_{\boldsymbol{A}^\pm}\|_{L^\infty}, \quad \textit{for all } t > 0.$$

Let us now discuss the multiplicity of solutions for (4.54). As seen before, when $\alpha = 1$, the self-similar solutions are explicitly given by

$$\boldsymbol{m}_{c,1}(x,t) = (\cos(c\,\mathrm{Erf}(x/\sqrt{t})), \sin(c\,\mathrm{Erf}(x/\sqrt{t})), 0), \qquad \text{for all } (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$

for all $c > 0$, where Erf is the non-normalized error function $\mathrm{Erf}(s) = \int_0^s e^{-\sigma^2/4}\, d\sigma$. In particular,

$$\vec{\boldsymbol{A}}^\pm_{c,1} = (\cos(c\sqrt{\pi}), \pm\sin(c\sqrt{\pi}), 0)$$

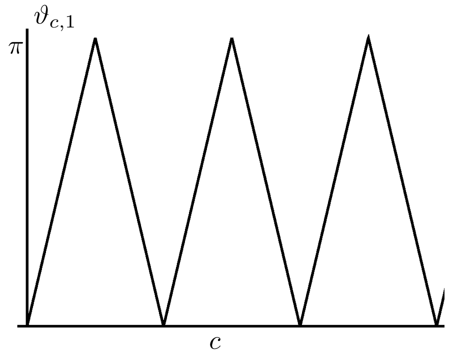


Figure 4.3: The angle $\vartheta_{c,\alpha}$ as a function of c for $\alpha = 1$.

and the angle between $\boldsymbol{A}^+_{c,1}$ and $\boldsymbol{A}^-_{c,1}$ is given by $\vartheta_{c,1} = \arccos(\cos(2c\sqrt{\pi}))$.

Figure 4.3 shows that there are infinite values of c that allow to reach any angle in $[0, \pi]$. Therefore, using the invariance of (4.1) under rotations, in the case when $\alpha = 1$, one can easily prove the existence of multiple solutions associated with a given initial data of the form $\boldsymbol{m}^0_{\boldsymbol{A}^\pm}$ for any given vectors $\boldsymbol{A}^\pm \in \mathbb{S}^2$. In the case that α is close enough to 1, we can use a continuity argument to prove that we still have multiple solutions. More precisely, we can establish that for any given initial data of the form $\boldsymbol{m}^0_{\boldsymbol{A}^\pm}$, with angle between $\boldsymbol{A}^+$ and $\boldsymbol{A}^-$ in the interval $(0, \pi)$, if α is sufficiently close to one, then there exist *at least* k-distinct solutions of (4.1) associated with the same initial condition, for any given $k \in \mathbb{N}$. In other words, given any angle $\vartheta \in (0, \pi)$ between two $\boldsymbol{A}^+$ and $\boldsymbol{A}^-$, we can generate any number of distinct solutions by considering values of α sufficiently close to 1. Precisely,

Theorem 4.24 ([dLG19])**.** *Let $k \in \mathbb{N}$, $\boldsymbol{A}^+$, $\boldsymbol{A}^- \in \mathbb{S}^2$ and let ϑ be the angle between $\boldsymbol{A}^+$ and $\boldsymbol{A}^-$. If $\vartheta \in (0, \pi)$, then there exists $\alpha_k \in (0, 1)$ such that for every $\alpha \in [\alpha_k, 1]$ there are at least k distinct smooth self-similar solutions $\{\boldsymbol{m}_j\}_{j=1}^k$ in $X(\mathbb{R} \times \mathbb{R}^+; \mathbb{S}^2)$ of (4.1) with initial condition $\boldsymbol{m}^0_{\boldsymbol{A}^\pm}$. These solutions are characterized by a strictly increasing sequence of values $\{c_j\}_{j=1}^k$, with $c_k \to \infty$ as $k \to \infty$, such that $\boldsymbol{m}_j = \mathcal{R}_j \boldsymbol{m}_{c_j,\alpha}$, where $\mathcal{R}_j \in SO(3)$. In particular*

$$\sqrt{t}\|\partial_x \boldsymbol{m}_j(\cdot, t)\|_{L^\infty(\mathbb{R})} = c_j, \quad \text{for all } t > 0. \tag{4.55}$$

Furthermore, if $\alpha = 1$ and $\vartheta \in [0, \pi]$, then there is an infinite number of distinct smooth self-similar solutions $\{\boldsymbol{m}_j\}_{j \geq 1}$ in $X(\mathbb{R} \times \mathbb{R}^+; \mathbb{S}^2)$ of (4.1) with initial condition $\boldsymbol{m}^0_{\boldsymbol{A}^\pm}$.

It is important to remark that in particular Theorem 4.24 asserts that when $\alpha = 1$, given $\boldsymbol{A}^+, \boldsymbol{A}^- \in \mathbb{S}^2$ such that $\boldsymbol{A}^+ = \boldsymbol{A}^-$, there exists an infinite number of distinct solutions $\{\boldsymbol{m}_j\}_{j \geq 1}$ in $X(\mathbb{R} \times \mathbb{R}^+; \mathbb{S}^2)$ of (4.1) with initial condition $\boldsymbol{m}^0_{\boldsymbol{A}^\pm}$ such that $[\boldsymbol{m}^0_{\boldsymbol{A}^\pm}]_{BMO} = 0$. This particular case shows that a condition on the size of X-norm of the solution as that given in (4.49) in Theorem 4.17 is necessary for the uniqueness of solution. We recall that for finite energy solutions of (HFHM) there are several nonuniqueness results based on Coron's technique [Cor90] in dimension $N = 3$. Alouges and Soyeur [AS92] successfully adapted this idea to prove the existence of multiple solutions of the (LLG$_\alpha$), with $\alpha > 0$, for maps $\boldsymbol{m} : \Omega \longrightarrow \mathbb{S}^2$, with Ω a bounded regular domain of $\mathbb{R}^3$. In our case, since $\{c_j\}_{j=1}^k$ is strictly increasing, we have at least k different *smooth* solutions. Notice also that the identity (4.55) implies that the X-norm of the solution is large as $j \to \infty$.

4.5 Shrinkers

We now discuss the backward self-similar solutions to (4.1), i.e. the shrinker solutions of the form
$$\boldsymbol{m}(x,t) = \boldsymbol{f}\left(\frac{x}{\sqrt{T-t}}\right), \qquad x \in \mathbb{R},\ t \in (-\infty, T).$$
As in Section 4.2, we can reduce our problem to the study of the ODE
$$\alpha \boldsymbol{f}'' + \alpha|\boldsymbol{f}'|^2\boldsymbol{f} + \beta(\boldsymbol{f} \times \boldsymbol{f}')' - \frac{x\boldsymbol{f}'}{2} = 0, \quad \text{ on } \mathbb{R}, \tag{4.56}$$
which is the same equation that we obtained for the expanders, expect for the minus sign in the last term. Following similar arguments, we get

Theorem 4.25 ([dLG20])**.** *Let $\alpha \in (0,1]$. Assume that $\boldsymbol{f} \in H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{S}^2)$ is a weak solution to* (4.4)*. Then $\boldsymbol{f}$ belongs to $\mathcal{C}^\infty(\mathbb{R};\mathbb{S}^2)$ and there exists $c \geq 0$ such that $|\boldsymbol{f}'(x)| = ce^{\alpha x^2/4}$, for all $x \in \mathbb{R}$. Moreover, the set of nonconstant solutions to* (4.56) *is $\{\mathcal{R}\boldsymbol{f}_{c,\alpha} : c > 0, \mathcal{R} \in SO(3)\}$, where is $\boldsymbol{f}_{c,\alpha}$ is given by the solution of the Serret–Frenet system*
$$\begin{aligned} \boldsymbol{f}' &= k\boldsymbol{g}, \\ \boldsymbol{g}' &= -k\boldsymbol{f} + \tau\boldsymbol{h}, \\ \boldsymbol{h}' &= -\tau\boldsymbol{g}. \end{aligned} \tag{4.57}$$
with curvature and torsion
$$k(x) = ce^{\frac{\alpha x^2}{4}} \qquad \text{and} \qquad \tau(x) = -\frac{\beta x}{2},$$
and initial conditions $\boldsymbol{f}(0) = (1,0,0)$, $\boldsymbol{g}(0) = (0,1,0)$, and $\boldsymbol{h}(0) = (0,0,1)$.

As done for the expanders, we provide now some properties of these solutions, that are obtained by studying the Serret–Frenet system (4.57).

Theorem 4.26 ([dLG20])**.** *Let $\alpha \in (0,1]$, $c > 0$, $T \in \mathbb{R}$ and $\boldsymbol{f}_{c,\alpha}$ as above. Set*
$$\tilde{\boldsymbol{m}}_{c,\alpha}(x,t) = \boldsymbol{f}_{c,\alpha}\left(\frac{x}{\sqrt{T-t}}\right), \qquad t < T. \tag{4.58}$$
Then we have the following statements.

(i) *The function $\tilde{\boldsymbol{m}}_{c,\alpha}$ belongs to $\mathcal{C}^\infty(\mathbb{R} \times (-\infty,T);\mathbb{S}^2)$, solves* (4.1) *for $t \in (-\infty,T)$, and*
$$|\partial_x \tilde{\boldsymbol{m}}_{c,\alpha}(x,t)| = \frac{c}{\sqrt{T-t}} e^{\frac{\alpha x^2}{4(T-t)}}, \quad \text{for all } (x,t) \in \mathbb{R} \times (-\infty,T).$$

(ii) *The profile* $\boldsymbol{f}_{c,\alpha} = (f_{1,c,\alpha}, f_{2,c,\alpha}, f_{3,c,\alpha})$ *satisfies that* $f_{1,c,\alpha}$ *is even, while* $f_{2,c,\alpha}$ *and* $f_{3,c,\alpha}$ *are odd.*

(iii) *There exist constants* $\rho_{j,c,\alpha} \in [0,1]$, $B_{j,c,\alpha} \in [-1,1]$, *and* $\phi_{j,c,\alpha} \in [0, 2\pi)$, *for* $j \in \{1,2,3\}$, *such that we have the following asymptotics for the profile* $\boldsymbol{f}_{c,\alpha}$*:*

$$\begin{aligned} f_{j,c,\alpha}(x) =& \rho_{j,c,\alpha} \cos(c\Phi_\alpha(x) - \phi_{j,c,\alpha}) - \frac{\beta B_{j,c,\alpha}}{2c} x e^{-\alpha x^2/4} \\ &+ \frac{\beta^2 \rho_{j,c,\alpha}}{8c} \sin(c\Phi_\alpha(x) - \phi_{j,c,\alpha}) \int_x^\infty s^2 e^{-\alpha s^2/4} ds + \frac{\beta}{\alpha^5 c^2} \mathcal{O}(x^2 e^{-\alpha x^2/2}), \end{aligned} \tag{4.59}$$

for all $x \geq 1$, *where*

$$\Phi_\alpha(x) = \int_0^x e^{\frac{\alpha s^2}{4}}\, ds.$$

Moreover, the constants satisfy the identities $\rho_{1,c,\alpha}^2 + \rho_{2,c,\alpha}^2 + \rho_{3,c,\alpha}^2 = 2$, $B_{1,c,\alpha}^2 + B_{2,c,\alpha}^2 + B_{3,c,\alpha}^2 = 1$ *and* $\rho_{j,c,\alpha}^2 + B_{j,c,\alpha}^2 = 1$, *for* $j \in \{1,2,3\}$*. In addition,*

$$\begin{aligned} f'_{j,c,\alpha}(x) &= -c\rho_{j,c,\alpha} \sin(c\Phi_\alpha(x) - \phi_{j,c,\alpha}) e^{\alpha x^2/4} \\ &+ \frac{\beta^2 \rho_{j,c,\alpha}}{8} \cos(c\Phi_\alpha(x) - \phi_{j,c,\alpha}) e^{\alpha x^2/4} \int_x^\infty s^2 e^{-\alpha s^2/4} ds + \frac{\beta}{\alpha^5 c} \mathcal{O}(x^2 e^{-\alpha x^2/4}), \end{aligned}$$

for all $x \geq 1$ *and* $j \in \{1,2,3\}$.

(iv) *The solution* $\tilde{\boldsymbol{m}}_{c,\alpha} = (\tilde{m}_{1,c,\alpha}, \tilde{m}_{2,c,\alpha}, \tilde{m}_{3,c,\alpha})$ *satisfies the following pointwise convergences*

$$\lim_{t \to T^-} (\tilde{m}_{j,c,\alpha}(x,t) - \rho_{j,c,\alpha} \cos\big(c\, \Phi_\alpha(\frac{x}{\sqrt{T-t}}) - \phi_{j,c,\alpha}\big) = 0, \ \text{if } x > 0,$$
$$\lim_{t \to T^-} (\tilde{m}_{j,c,\alpha}(x,t) - \rho^-_{j,c,\alpha} \cos\big(c\, \Phi_\alpha(\frac{-x}{\sqrt{T-t}}) - \phi_{j,c,\alpha}\big) = 0, \ \text{if } x < 0,$$

for $j \in \{1,2,3\}$, *where* $\rho^-_{1,c,\alpha} = \rho_{1,c,\alpha}$, $\rho^-_{2,c,\alpha} = -\rho_{2,c,\alpha}$ *and* $\rho^-_{3,c,\alpha} = -\rho_{3,c,\alpha}$.

(v) *For any* $\varphi \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^3)$, *we have*

$$\lim_{t \to T^-} \int_{\mathbb{R}} \tilde{\boldsymbol{m}}_{c,\alpha}(x,t) \cdot \varphi(x) dx = 0.$$

In particular $\tilde{\boldsymbol{m}}_{c,\alpha}(\cdot, t) \to 0$ *as* $t \to T^-$, *as a tempered distribution.*

Let us first recall that the existence of smooth solutions to (4.1) on short times can be established as in the case of the heat flow for harmonic maps [LW08], using that (4.1) is a strongly parabolic system [GH93, Ama86]. In particular, in the one-dimensional case, for any initial condition $\boldsymbol{m}^0 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{S}^2)$, there exists a maximal time $0 < T_{\max} \leq \infty$ such that (4.1) admits a unique, smooth solution $\boldsymbol{m} \in \mathcal{C}^\infty(\mathbb{R} \times [0, T_{\max}); \mathbb{S}^2)$. Moreover, if $T_{\max} < \infty$, then

$$\lim_{t \to T^-_{\max}} \|\partial_x \boldsymbol{m}(\cdot, t)\|_{L^\infty(\mathbb{R})} = \infty.$$

Therefore, for any $c > 0$ and $T \in \mathbb{R}$, taking (4.1) with initial condition at time $T-1$

$$\boldsymbol{m}^0(\cdot, T-1) := \boldsymbol{f}_{c,\alpha}(\cdot),$$

the solution is given by $\tilde{\boldsymbol{m}}_{c,\alpha}$ in Theorem 4.26, for $t \in [T-1, T)$, and blows up at time T. Indeed, from (i), for $c > 0$ and for all $x \in \mathbb{R}$,

$$\lim_{t \to T^-} |\partial_x \tilde{\boldsymbol{m}}_{c,\alpha}(x,t)| = \lim_{t \to T^-} \frac{c}{\sqrt{T-t}} e^{\frac{\alpha x^2}{4(T-t)}} = \infty.$$

Part (iii) of the above theorem provides the asymptotics of the profile $\tilde{\boldsymbol{m}}_{c,\alpha}$ at infinity, in terms of a fast oscillating principal part, plus some exponentially decaying terms. Notice that for the integral term in (4.59), we have (see e.g. [AS64])

$$\int_x^\infty s^2 e^{-\alpha s^2/4} ds \sim \frac{2xe^{-\alpha x^2/4}}{\alpha}\Big(1 + \frac{2}{\alpha x^2} - \frac{4}{\alpha^2 x^4} + \cdots\Big), \qquad \text{as } x \to \infty.$$

It is also important to mention in the asymptotics (4.59), the big-$\mathcal{O}$ does not depend on the parameters, i.e. there exists a universal constant $C > 0$, such that the big-$\mathcal{O}$ in (4.59) satisfies

$$|\mathcal{O}(x^2 e^{-\alpha x^2/2})| \le C x^2 e^{-\alpha x^2/2}, \qquad \text{for all } x \ge 1.$$

In this manner, the constants multiplying the big-$\mathcal{O}$ are meaningful and in particular, big-$\mathcal{O}$ vanishes when $\beta = 0$ (i.e. $\alpha = 1$). Let us remark that the behavior of the profile for $x \ge -1$ follows from the symmetries of the profile established in part (ii). Finally, by plugging (4.59) in (4.58), we obtain a precise description of the fast oscillating nature of the blow up of the solution $\tilde{\boldsymbol{m}}_{c,\alpha}$.

In Figure 4.4 we have depicted the profile $\tilde{\boldsymbol{m}}_{c,\alpha}$ for $\alpha = 0.5$ and $c = 0.5$, where we can see this oscillating behavior. Moreover, the plots in Figure 4.4 suggest that the limit sets of the trajectories are great circles on the sphere $\mathbb{S}^2$ when $x \to \pm\infty$. This is indeed the case. In our last result we establish analytically that $\tilde{\boldsymbol{m}}_{c,\alpha}$ oscillates in a plane passing through the origin whose normal vector is given by $\boldsymbol{B}^\pm_{c,\alpha} = (B_{1,c,\alpha}, B_{2,c,\alpha}, B_{3,c,\alpha})$, as $x \to \pm\infty$, respectively.

Theorem 4.27 ([dLG19]). *Using the constants given in Theorem 4.26, let $\mathcal{P}^\pm_{c,\alpha}$ be the planes passing through the origin with normal vectors $\boldsymbol{B}^\pm_{c,\alpha}$, respectively. Let $\mathcal{C}^\pm_{c,\alpha}$ be the circles in $\mathbb{R}^3$ given by the intersection of these planes with the sphere, i.e. $\mathcal{C}^\pm_{c,\alpha} = \mathcal{P}^\pm_{c,\alpha} \cap \mathbb{S}^2$. Then the following statements hold.*

(i) *For all $|x| \ge 1$, we have*

$$\operatorname{dist}(\tilde{\boldsymbol{m}}_{c,\alpha}(x), \mathcal{C}^\pm_{c,\alpha})) \le \frac{15\sqrt{2}\beta}{c\alpha^2} |x| e^{-\alpha x^2/4}. \tag{4.60}$$

In particular

$$\lim_{t \to T^-} \operatorname{dist}(\tilde{\boldsymbol{m}}_{c,\alpha}(x,t), \mathcal{C}^+_{c,\alpha})) = 0, \text{ if } x > 0, \text{ and } \lim_{t \to T^-} \operatorname{dist}(\tilde{\boldsymbol{m}}_{c,\alpha}(x,t), \mathcal{C}^-_{c,\alpha})) = 0, \text{ if } x < 0. \tag{4.61}$$

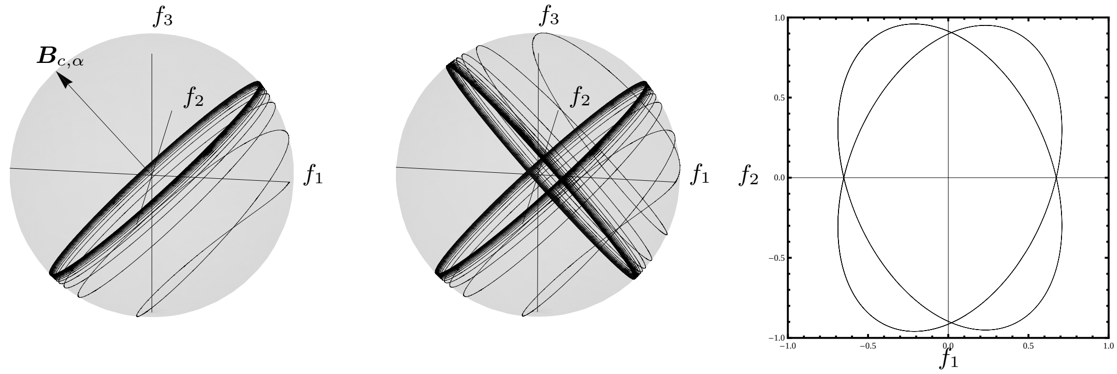
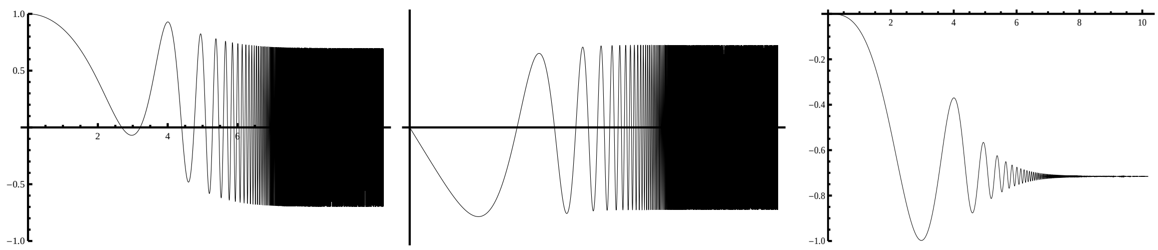


Figure 4.4: Profile $\boldsymbol{f}_{c,\alpha}$ for $c = 0.5$ and $\alpha = 0.5$. The figure on the left depicts profile for $x \in \mathbb{R}^+$ and the normal vector $\boldsymbol{B}_{c,\alpha} \approx (-0.72, -0.3, 0.63)$. The figure on the center shows the profile for $x \in \mathbb{R}$; the angle between the circles $\mathcal{C}^{\pm}_{c,\alpha}$ is $\vartheta_{c,\alpha} \approx 1.5951$. At the right, the projection of limit cycles $\mathcal{C}^{\pm}_{c,\alpha}$ on the plane $\mathbb{R}^2$.



(a) $f_{1,c,\alpha}$ (b) $g_{1,c,\alpha}$ (c) $g_{1,c,\alpha}$

Figure 4.5: Functions $f_{1,c,\alpha}$, $g_{1,c,\alpha}$ and $h_{1,c,\alpha}$ for $c = 0.5$ and $\alpha = 0.5$ on $\mathbb{R}^+$. The limit at infinity in (iii) is $B_{1,c,\alpha} \approx -0.72$.

(ii) *Let* $\vartheta_{c,\alpha} = \arccos(1 - 2B^2_{1,c,\alpha})$ *be the angle between the circles* $\mathcal{C}^{\pm}_{c,\alpha}$. *For* $c \geq \beta\sqrt{\pi}/\sqrt{\alpha}$, *we have*

$$\vartheta_{c,\alpha} \geq \arccos\left(-1 + \frac{2\pi\beta^2}{c^2\alpha}\right). \tag{4.62}$$

In particular

$$\lim_{c\to\infty} \vartheta_{c,\alpha} = \pi, \quad \text{for all } \alpha \in (0,1], \quad \textit{and} \quad \lim_{\alpha\to 1} \vartheta_{c,\alpha} = \pi, \quad \text{for all } c > 0.$$

Theorem 4.27 establishes the convergence of the profile $\boldsymbol{f}_{c,\alpha}$ to the great circles $\mathcal{C}^{\pm}_{c,\alpha}$ as shown in Figure 4.4. Moreover, (4.60) gives us an exponential rate for this convergence. In terms of the solution $\tilde{\boldsymbol{m}}_{c,\alpha}$ to the LLG equation, this provides a more precise geometric information about the way that the solution blows up at time T, as seen in (4.61). The existence of limit cycles for related ferromagnetic models have been investigated for instance in [WBY85, BMSB87] but to the best of our knowledge, this is the first time that this type of phenomenon has been observed for the LLG equation. In Figure 4.4 can see that $\vartheta_{c,\alpha} \approx 1.5951$

for $\alpha = 0.5$ and $c = 0.5$, where we have chosen the value of c such that the angle is close to $\pi/2$.

In addition, (4.62) in Theorem 4.27 provides some geometric information about behavior of the limit circles with respect to the parameters c and α. In particular, we deduce that they converge to the same plane as $c \to \infty$, for fixed $\alpha \in (0, 1]$, and the same happens as $\alpha \to 1$, for fixed $c > 0$.

4.5.1 Comparison with the limit cases

In the case $\alpha = 1$, the torsion vanishes, and it easy to deduce that the profile is explicitly given by the plane curve

$$\boldsymbol{f}_{c,1}(x) = (\cos(c\Phi_1(x)), \sin(c\Phi_1(x)), 0).$$

In particular, we see that the asymptotics in Theorem 4.26 are satisfied with $\rho_{1,c,1} = 1$, $\rho_{2,c,1} = 1$, $\rho_{3,c,1} = 0$, $\phi_{1,c,1} = 0$, $\phi_{2,c,1} = 3\pi/2$, $\phi_{3,c,1} \in [0, 2\pi)$.

As mentioned in Subsection 4.2.1, the case $\alpha = 0$ is more involved, but the solution $\{\boldsymbol{f}_{c,0}, \boldsymbol{g}_{c,0}, \boldsymbol{h}_{c,0}\}$ of the system (16) can still be explicitly determined in terms of confluent hypergeometric functions, using that the curvature is constant $k(x) = c$, for all $x \in \mathbb{R}$. This leads to the asymptotics [GRV03, dLG15b, GL]

$$\boldsymbol{f}_{c,0}(x) = \boldsymbol{A}_c - \frac{2c}{x}\boldsymbol{B}_c \cos\left(\frac{x^2}{4} + c^2 \ln(x)\right) + O\left(\frac{1}{x^2}\right), \tag{4.63}$$

as $x \to \infty$, for some vectors $\boldsymbol{A}_c \in \mathbb{S}^2$ and $\boldsymbol{B}_c \in \mathbb{R}^3$. In particular, we see that $\boldsymbol{f}_{c,0}$ converges to the point $\boldsymbol{A}_c$, as $x \to \infty$. Hence, there is a drastic change in the behavior of the profile in the cases $\alpha = 0$ and $\alpha > 0$: In the first case $\boldsymbol{f}_{c,0}$ converges to a point at infinity, while in the second case (4.60) tells us that $\boldsymbol{f}_{c,\alpha}$ converges to a great circle. In this sense, there is a discontinuity in the behavior of $\boldsymbol{m}_{c,\alpha}$ at $\alpha = 0$.

Going back to Theorem 4.26, it is seems very difficult to get asymptotics for the constants in (4.59). As explained before, our strategy for the constants appearing in the asymptotics for the expanders relied on obtaining uniform estimates and using continuity arguments. In particular, using the fact that the constants in (4.63) are explicit, we were able to get good information about the constants in the asymptotics when α was close to 0. Due to the above mentioned discontinuity of $\boldsymbol{f}_{c,\alpha}$ at $\alpha = 0$, it seems unlike to use this argument in the asymptotics for the shrinkers.

Finally, let us also remark that we cannot use continuation arguments to find the behavior of the circles for c small. This is expected since for $c = 0$, the explicit solution to Serret–Frenet system is given by

$$\begin{aligned}
\boldsymbol{f}_{0,\alpha}(x) &= (1, 0, 0),\\
\boldsymbol{g}_{0,\alpha}(x) &= (0, \cos(\beta x^2/4), -\sin(\beta x^2/4)),\\
\boldsymbol{h}_{0,\alpha}(x) &= (0, \sin(\beta x^2/4), \cos(\beta x^2/4)).
\end{aligned}$$

for all $x \in \mathbb{R}$.

4.5.2 Ideas of the proofs

The asymptotics in Theorem 4.26 are simpler to obtain than the ones in Theorem 4.4, since we do not have to keep uniform estimates in α. For simplicity, we will drop the subscripts c and α, if there is no possible confusion. Taking into account the block structure of the nine equation in the Serret–Frenet system (4.57), it suffices to analyze the following system of three equations:

$$\begin{cases} f' = ce^{\alpha x^2/4} g, \\ g' = -ce^{\alpha x^2/4} f - \dfrac{\beta x}{2} h, \\ h' = \dfrac{\beta x}{2} g. \end{cases} \tag{4.64}$$

Integrating h', we get

$$h(x) - h(0) == \frac{\beta}{2c}\int_0^x se^{-\frac{\alpha s^2}{4}} f'(s)ds = \frac{\beta}{2c}\Big(xe^{-\frac{\alpha x^2}{4}} f(x) - \int_0^x \big(1 - \frac{\alpha s^2}{2}\big)e^{-\frac{\alpha s^2}{4}} f(s)ds\Big), \tag{4.65}$$

where we have used integration by parts. Notice that $\int_0^\infty (1-\alpha s^2/2)e^{-\alpha s^2/4} f(s)\, ds$ is well-defined, since $\alpha \in (0,1]$ and $|f| \leq 1$. Therefore, the existence of $B := \lim_{x\to\infty} h(x)$ follows from (4.65). Moreover,

$$B := h(0) - \frac{\beta}{2c}\int_0^\infty \left(1 - \frac{\alpha s^2}{2}\right) e^{-\alpha s^2/4} f(s)ds.$$

Integrating again h' in (4.64) from $x \in \mathbb{R}$ to ∞, and arguing as above, we conclude that

$$h(x) = B + \frac{\beta x}{2c} e^{-\alpha x^2/4} f(x) + \frac{\beta}{2c}\int_x^\infty \left(1 - \frac{\alpha s^2}{2}\right) e^{-\alpha s^2/4} f(s)ds,$$

and we can deduce that for all $x \geq 1$

$$|h(x) - B| \leq \frac{6\beta}{c\alpha} xe^{-\alpha x^2/4}.$$

On the other hand, setting $w = f + ih$ and using (4.64), we obtain that w satisfies the ODE

$$w' + ice^{\alpha x^2/4} w = -i\frac{\beta x}{2} h(x).$$

Multiplying by the integrating factor $ic\Phi_\alpha$, and integrating by parts using that $ice^{ic\Phi_\alpha(x)} = (e^{ic\Phi_\alpha(x)})' e^{-\alpha x^2/4}$, we finally get

$$e^{ic\Phi_\alpha} w(x) = w(0) - \frac{\beta x h(x)}{2c} e^{ic\Phi_\alpha(x) - \alpha x^2/4} + \frac{\beta}{2c}\int_0^x e^{ic\Phi_\alpha(s) - \alpha s^2/4}\big(\frac{\beta}{2}s^2 g(s) + (1 - \alpha\frac{s^2}{2})h(s)\big)ds.$$

Since $\alpha \in (0,1]$, from the above identity if follows the existence of the limit

$$W := \lim_{x\to\infty} e^{ic\Phi_\alpha(x)} w(x),$$

and the exponential rate

$$|w(x) - e^{-ic\,\Phi_\alpha(x)}W| \leq \frac{10\beta}{c\alpha^2} x e^{-\alpha x^2/4},$$

for all $x \geq 1$. Using again (4.64), and after several integrations by parts, we finally obtain the asymptotics in part (iii) in Theorem 4.26. Parts (iv) and ((v)) are straightforward consequences of these asymptotics.

The proof of Theorem 4.27 is based on the following formula.

Lemma 4.28. *The constants given by Theorem 4.26 satisfy the following identity*

$$B_{1,c,\alpha}\rho_{1,c,\alpha}e^{i\phi_{1,c,\alpha}} + B_{2,c,\alpha}\rho_{2,c,\alpha}e^{i\phi_{2,c,\alpha}} + b_{3,\infty}\rho_{3,c,\alpha}e^{i\phi_{3,c,\alpha}} = 0. \tag{4.66}$$

Indeed, using again the asymptotics in Theorem 4.26 and that $(\boldsymbol{f} + i\boldsymbol{g}) \cdot \boldsymbol{h} = 0$, we have

$$h_1(x)\rho_1 e^{-i(c\Phi_\alpha(x)-\phi_1)} + h_2(x)\rho_2 e^{-i(c\Phi_\alpha(x)-\phi_2)} + h_3(x)\rho_3 e^{-i(c\Phi_\alpha(x)-\phi_3)} = o(1),$$

so that we deduce (4.66) dividing by $e^{-ic\Phi_\alpha(x)}$ and letting $x \to \infty$.

To prove (4.60), the first step is to estimate the distance between the profile $\tilde{\boldsymbol{m}}$ and the plane $\mathcal{P}^+$. Since B is the normal vector of this plane and $|B_{c,\alpha}| = 1$, we have

$$\operatorname{dist}(\tilde{\boldsymbol{m}}(x), \mathcal{P}^+) = |\tilde{m}_1(x)B_1 + \tilde{m}_2(x)B_2 + \tilde{m}_3(x)B_3|.$$

To compute the leading term, using (4.66), we conclude that

$$\sum_{j=1}^{3} \rho_j \cos(c\Phi_\alpha(x - \phi_j) = \operatorname{Re}\left(e^{ic\Phi_\alpha(x)}\left(\sum_{j=1}^{3} \rho_j B_j e^{i\phi_j}\right)\right) = 0.$$

Thus, the asymptotics in Theorem 4.26 give us, for $x \geq 1$,

$$\operatorname{dist}(\tilde{\boldsymbol{m}}(x), \mathcal{P}^+)) \leq \frac{10\beta}{c\alpha^2} x e^{-\alpha x^2/4}(\rho_1 B_1 + \rho_2 B_2 + \rho_3 B_3) \leq \frac{15\beta}{c\alpha^2} x e^{-\alpha x^2/4}, \tag{4.67}$$

where we have used that

$$\rho_j B_j \leq \frac{1}{2}(\rho_j^2 + B_j^2) = \frac{1}{2}.$$

Finally, (4.60) follows from estimate (4.67), by using some elementary geometric arguments.

Chapter 5

The nonlocal Gross–Pitaevskii equation

In this final chapter we consider a nonlocal family of Gross–Pitaevskii equations with nonzero conditions at infinity, in the one dimensional case. We present here a result in collaboration with P. Mennuni [dLM20], that provides conditions on the nonlocal interaction such that there is a branch of traveling waves solutions with nonvanishing conditions at infinity. Moreover, we show that the branch is orbitally stable. In this manner, this result generalizes known properties for the contact interaction given by a Dirac delta function.

As a by-product of our analysis, we provide a simple condition to ensure that the solution to the Cauchy problem is global in time, improving a previous result in [dL10].

5.1 The nonlocal equation

In order to describe the dynamics of a weakly interacting Bose gas of bosons of mass m, Gross [Gro63] and Pitaevskii [Pit61] derived in the Hartree approximation, that the wavefunction Ψ governing the condensate satisfies

$$i\hbar\partial_t\Psi(x,t) = -\frac{\hbar^2}{2m}\Delta\Psi(x,t) + \Psi(x,t)\int_{\mathbb{R}^N}|\Psi(y,t)|^2V(x-y)\,dy, \text{ on } \mathbb{R}^N\times\mathbb{R}, \tag{5.1}$$

where V describes the interaction between bosons. In the most typical first approximation, V is considered as a Dirac delta function, which leads to the standard local Gross–Pitaevskii equation. This local model with nonvanishing condition at infinity has been intensively used, due to its application in various areas of physics, such as superfluidity, nonlinear optics and Bose-Einstein condensation [JR82, JPR86, KL98, Cos98]. It seems then natural to analyze equation (5.1) for more general interactions. Indeed, in the study of superfluidity, supersolids and Bose-Einstein condensation, different types of nonlocal potentials have been proposed [DK03, SK04, ABJ07, YY01, CMKF09].

To obtain a dimensionless equation, we take the average energy level per unit mass E_0 of a boson, and we set

$$\psi(x,t) = \exp\left(\frac{imE_0 t}{\hbar}\right)\Psi(x,t).$$

Then (5.1) turns into

$$i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\Delta\psi(x,t) - mE_0\psi(x,t) + \psi(x,t)\int_{\mathbb{R}^N}|\psi(y,t)|^2 V(x-y)\,dy. \tag{5.2}$$

Defining the rescaling

$$u(x,t) = \frac{1}{\lambda\sqrt{mE_0}}\left(\frac{\hbar}{\sqrt{2m^2E_0}}\right)^{\frac{N}{2}}\psi\left(\frac{\hbar x}{\sqrt{2m^2E_0}}, \frac{\hbar t}{mE_0}\right),$$

from (5.2) we deduce that

$$i\partial_t u(x,t) + \Delta u(x,t) + u(x,t)\left(1 - \lambda^2\int_{\mathbb{R}^N}|u(y,t)|^2\mathcal{V}(x-y)\,dy\right) = 0,$$

with

$$\mathcal{V}(x) = V\left(\frac{\hbar x}{\sqrt{2m^2E_0}}\right).$$

If we assume that the convolution between $\mathcal{V}$ and a constant is well-defined and equal to a positive constant, choosing $\lambda^2 = (\mathcal{V} * 1)^{-1}$, equation (5.2) is equivalent to

$$i\partial_t u + \Delta u + \lambda^2 u(\mathcal{V} * (1-|u|^2)) = 0 \text{ on } \mathbb{R}^N \times \mathbb{R}.$$

More generally, we consider the nonlocal Gross–Pitaevskii equation with nonzero initial condition at infinity in the form

$$\begin{cases} i\partial_t u + \Delta u + u(\mathcal{W} * (1-|u|^2)) = 0 \text{ on } \mathbb{R}^N \times \mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{NGP}$$

where $|u_0(x)| \to 1$, as $|x| \to \infty$.

If $\mathcal{W}$ is a real-valued even distribution, (NGP) is a Hamiltonian equation whose energy given by

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u(t)|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^N}(\mathcal{W} * (1-|u(t)|^2))(1-|u(t)|^2)\,dx$$

is formally conserved.

5.2 The Cauchy problem

In the case that $\mathcal{W}$ is the Dirac delta function, (NGP) corresponds to the local Gross–Pitaevskii equation and the Cauchy problem in this instance has been studied in [BS99], Gérard [Gér06], Gallo [Gal08b], among others. During my Ph.D. thesis, I gave sufficient conditions, covering a variety of nonlocal interactions, such that the associated Cauchy problem was globally well-posed, in any dimension. For the sake of simplicity, we recall here only the results dimensions $1 \leq N \leq 3$.

To state main result in [dL10], we first recall that for a tempered distribution $\mathcal{V} \in S'(\mathbb{R})$, we can define the convolution with a function in $L^p(\mathbb{R})$, with $1 \leq p < \infty$, through the Fourier transform, as the bounded extension on $L^p(\mathbb{R})$ of the operator

$$\mathcal{V} * f := \mathcal{F}^{-1}(\widehat{\mathcal{V}}\ \hat{f}), \quad f \in S(\mathbb{R}).$$

In this manner, the set

$$\mathcal{M}_p(\mathbb{R}) = \{\mathcal{V} \in S'(\mathbb{R}) : \exists C > 0, \|\mathcal{V} * f\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})}, \forall f \in L^p(\mathbb{R})\}$$

is a Banach space endowed with the operator norm denoted by $\|\cdot\|_{\mathcal{M}_p}$. Thus $\widehat{\mathcal{V}} \in \mathcal{M}_2(\mathbb{R}^N)$ is equivalent to $\mathcal{V} \in L^\infty(\mathbb{R}^N)$, with $\|\widehat{\mathcal{V}}\|_{L^\infty(\mathbb{R})} = \|\mathcal{V}\|_{\mathcal{M}_2}$. In addition, these spaces satisfy that $\mathcal{M}_1(\mathbb{R}^N) \subseteq \mathcal{M}_p(\mathbb{R}^N) \subseteq \mathcal{M}_2(\mathbb{R}^N)$ for all $1 \leq p \leq 2$. We refer to [Gra08] for further details about the properties of $\mathcal{M}_p(\mathbb{R})$.

Two examples to bear in mind are Dirac delta δ_0 and the integrable functions. Indeed, since δ_0 is the identity for the convolution, it belongs to $\mathcal{M}_p(\mathbb{R}^N)$, for all $p \in [1, \infty)$. The fact that $L^1(\mathbb{R}^N) \subset \mathcal{M}_p(\mathbb{R}^N)$, for all $p \in [1, \infty)$, follows from the Young inequality.

We are interested in studying the dynamics of (NGP), allowing different potentials $\mathcal{W}$. For this reason, we set the following energy space, independent of $\mathcal{W}$,

$$\mathcal{E}(\mathbb{R}) = \{v \in H^1_{\mathrm{loc}}(\mathbb{R}^{\mathbb{N}}) : 1 - |v|^2 \in L^2(\mathbb{R}^{\mathbb{N}}),\ \nabla v \in L^2(\mathbb{R}^{\mathbb{N}})\}.$$

Then the energy

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\, dx + \frac{1}{4}\int_{\mathbb{R}^N} (\mathcal{W} * (1 - |v|^2))(1 - |v|^2)\, dx$$

is finite, for all $(v, \mathcal{W}) \in \mathcal{E}(\mathbb{R}^N) \times \mathcal{M}_2(\mathbb{R}^N)$.

Theorem 5.1 ([dL10])**.** *Let $1 \leq N \leq 3$, $\phi_0 \in \mathcal{E}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, with $\nabla\phi \in H^\infty(\mathbb{R}^N)$. Let $\mathcal{W} \in \mathcal{M}_2(\mathbb{R}^N) \cap \mathcal{M}_3(\mathbb{R}^N)$ be an even distribution. Assume that one of the following is satisfied.*

(i) *$\mathcal{W} \in \mathcal{M}_1(\mathbb{R}^N)$ and $\mathcal{W} \geq 0$ in a distributional sense.*

(ii) *There exists $\sigma > 0$ such that $\widehat{\mathcal{W}} \geq \sigma$ a.e. on $\mathbb{R}^N$.*

(iii) *$N \geq 2$ and $\widehat{\mathcal{W}} \geq 0$ a.e. on $\mathbb{R}^N$.*

Then, for every $w_0 \in H^1(\mathbb{R}^N)$ *there exists a unique solution* $\Psi \in C(\mathbb{R}, \phi_0 + H^1(\mathbb{R}))$ *to* (NGP) *with the initial condition* $\Psi_0 = \phi_0 + w_0$*. Moreover, the energy is conserved, as well as the momentum as long as* $\inf_{x\in\mathbb{R}} |\Psi(x,t)| > 0$*.*

In the case (ii)*, we also have the growth estimate*

$$\|\Psi(t) - \phi_0\|_{L^2(\mathbb{R}^N)} \leq C|t| + \|\Psi_0 - \phi_0\|_{L^2(\mathbb{R}^N)}, \tag{5.3}$$

for any $t \in \mathbb{R}$*, where* $C > 0$ *is a constant depending only on* $E(\Psi_0)$*,* $\|\widehat{\mathcal{W}}\|_{L^\infty}$*,* ϕ_0 *and* σ*.*

In order to show Theorem 5.1, we first proved the local well-posedness. This was based on the fact that (NGP) can be recast as $u = w + \phi_0$, where w solves

$$\begin{cases} i\partial_t w + \Delta w + f(w) = 0 \text{ on } \mathbb{R}^N \times \mathbb{R}, \\ \qquad\qquad w(0) = w_0, \end{cases} \tag{5.4}$$

with $f(w) = g_1(w) + g_2(w) + g_3(w) + g_4(w)$, where

$$\begin{aligned} g_1(w) &= \Delta\phi + (W * (1 - |\phi|^2))\phi, \\ g_2(w) &= -2(W * \langle \phi, w\rangle)\phi, \\ g_3(w) &= -(W * |w|^2)\phi - 2(W * \langle\phi, w\rangle)w + (W * (1 - |\phi|^2))w, \\ g_4(w) &= -(W * |w|^2)w. \end{aligned}$$

Then the local existence was proved using Strichartz estimates and the Banach fixed-point theorem. This argument provided a local solution on a maximal interval $(-T_{\min}, T_{\max})$, as well as blow-up alternative for the norm $\|w(t)\|_{H^1(\mathbb{R}^N)}$. Using the conservation of the energy, deduced that $\sup\{\|\nabla w(t)\|_{L^2(\mathbb{R}^N)} : t \in (-T_{\min}, T_{\max})\}$ remained bounded, so that it was sufficient to show that

$$\sup\{\|w(t)\|_{L^2(\mathbb{R}^N)} : t \in (-T_{\min}, T_{\max})\} < \infty, \tag{5.5}$$

to conclude that the solution was global.

Let us recall the argument in the case ((ii)), that uses an idea introduced in [BV08] for the local Gross–Pitaevskii equation. By invoking the Plancherel theorem and the conservation of energy, we have

$$\sigma\||\phi + w(t)|^2 - 1\|^2_{L^2(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} (\mathcal{W} * (|\phi + w(t)|^2 - 1))(|\phi + w(t)|^2 - 1)\, dx \leq 4E_0, \tag{5.6}$$

where $E_0 = E(\phi_0 + w_0)$. On the other hand, from (5.4),

$$\begin{aligned} \frac{1}{2}\left|\frac{d}{dt}\|w(t)\|^2_{L^2}\right| &= \left|\operatorname{Im}\int_{\mathbb{R}^N} (\Delta\phi + \phi(\mathcal{W} * (1 - |\phi + w(t)|^2))\overline{w}(t)\, dx.\right| \\ &\leq \|\Delta\phi\|_{L^2}\|w(t)\|_{L^2} + \|\widehat{\mathcal{W}}\|_{L^\infty}\|\phi\|_{L^\infty}\||\phi + w(t)|^2 - 1\|_{L^2}\|w(t)\|_{L^2}. \end{aligned} \tag{5.7}$$

By combining (5.6) and (5.7), and integrating between 0 and t, we infer the linear growth

$$\|w(t)\|_{L^2} \leq \left(\|\Delta\phi\|_{L^2} + \|\widehat{\mathcal{W}}\|_{L^\infty} \|\phi\|_{L^\infty} \sqrt{\frac{4E_0}{\sigma}} \right) |t| + \|w_0\|_{L^2},$$

for any $t \in (-T_{\min}, T_{\max})$, which implies the global existence of the solution.

The proof of (5.5) in the case (iii) uses the Gagliardo–Nirenberg inequality

$$\|f\|^2_{L^4(\mathbb{R}^N)} \leq C\|\nabla f\|^{N/2}_{L^2} \|f\|^{2-N/2}_{L^2},$$

which gives, after some computations,

$$\frac{1}{2}\left|\frac{d}{dt}\|w(t)\|^2_{L^2}\right| \leq C(E_0, \mathcal{W})(1 + \|w(t)\|^2_{L^2} + \|w(t)\|^{3-N/2}_{L^2}).$$

Therefore we can conclude, for instance using a Gronwall argument, that $\|w(t)\|_{L^2}$ remains bounded iff $3 - N/2$, i.e. $N \geq 2$.

In the one-dimensional case, we have introduced in [dLM20] a condition to improve Theorem 5.1 in dimension $N = 1$, based on the following estimate.

Lemma 5.2 ([dLM20])**.** *Assume that $\mathcal{W} \in \mathcal{M}_2(\mathbb{R})$ satisfies*

$$\widehat{\mathcal{W}}(\xi) \geq (1 - \kappa\xi^2)^+, \quad \text{a.e. on } \mathbb{R}, \tag{5.8}$$

for some $\kappa \geq 0$. Let $v \in \mathcal{E}(\mathbb{R})$ and set $\eta := 1 - |v|^2$. Then

$$\|\eta\|^2_{L^\infty} + \|\eta\|^2_{L^2} \leq 16\tilde{\kappa}E(v)(1 + 8\tilde{\kappa}E(v) + 2\sqrt{2\tilde{\kappa}E(v)}),$$

with $\tilde{\kappa} = \kappa + 1$.

The basic idea is that for $v \in \mathcal{E}(\mathbb{R})$, setting $\eta = 1 - |v|^2$ and using the Plancherel identity, we get

$$\eta^2(x) = 2\int_{-\infty}^x \eta\eta' \leq \int_{\mathbb{R}} (\eta^2 + \eta'^2) = \frac{1}{2\pi}\int_{\mathbb{R}} (1 + \xi^2)|\hat{\eta}|^2 d\xi. \tag{5.9}$$

By (5.8), we have $1 \leq \widehat{\mathcal{W}}(\xi) + \kappa\xi^2$ a.e. on $\mathbb{R}$, so that the term on the right-hand side of (5.9) can be bounded by

$$\frac{1}{2\pi}\int_{\mathbb{R}} (1 + \xi^2)|\hat{\eta}|^2 \leq \frac{1}{2\pi}\int_{\mathbb{R}} (\widehat{\mathcal{W}}(\xi) + \tilde{\kappa}\xi^2)|\hat{\eta}|^2 = 4E(v) + \tilde{\kappa}\int_{\mathbb{R}} \eta'^2.$$

The rest of the proof consists in showing that $\|\eta'\|_{L^2(\mathbb{R})}$ can be also bounded in terms of the energy.

Theorem 5.3 ([dLM20])**.** *Let ϕ_0 and $\mathcal{W}$ be as in Theorem 5.1, but instead of (i), (ii) or (ii), we assume that there exists $\kappa \geq 0$ such that*

$$\widehat{\mathcal{W}}(\xi) \geq (1 - \kappa\xi^2)^+, \quad \text{a.e. on } \mathbb{R}.$$

Then we have the same conclusion as in Theorem 5.3, including the growth estimate (5.3)*, with a constant C depending only on $E(\Psi_0)$, $\|\widehat{\mathcal{W}}\|_{L^\infty}$, ϕ_0 and κ.*

Let us remark if $\widehat{\mathcal{W}} \in L^\infty(\mathbb{R})$ is even and of class C^2 in a neighbor of the origin, with $\widehat{\mathcal{W}} \geq 0$ a.e. on $\mathbb{R}$, then (5.8) is satisfied.

We discuss now some examples such that Theorems 5.1 and 5.3 guarantee global existence.

(i) We consider a generalization of the model proposed by Shchesnovich and Kraenkel [SK04]

$$\widehat{\mathcal{W}}(\xi) = \frac{1}{(1+a|\xi|^2)^{b/2}}, \quad a, b > 0,$$

We have $\widehat{\mathcal{W}} \in L^\infty(\mathbb{R}^N)$ and $\mathcal{W} \in L^1(\mathbb{R}^N)$, provide that $b > N-1$. Moreover, $\widehat{\mathcal{W}}$ is of class C^2 if $b > 1$. Therefore, under these conditions, we have global existence.

(ii) For $\beta > 2\alpha > 0$, we consider $\mathcal{W}_{\alpha,\beta} = \frac{\beta}{\beta-2\alpha}(\delta_0 - \alpha e^{-\beta|x|})$, so its Fourier transform is

$$\widehat{\mathcal{W}}_{\alpha,\beta}(\xi) = \frac{\beta}{\beta-2\alpha}\Big(1 - \frac{2\alpha\beta}{|\xi|^2+\beta^2}\Big),$$

and thus we also have global existence for this potential.

We now recall the following results about the energy space $\mathcal{E}(\mathbb{R}^N)$, for $1 \leq N \leq 3$; we refer to [Gér06, G ́, Gal08b] for their proofs.

Lemma 5.4. *Let $u \in \mathcal{E}(\mathbb{R}^N)$. Then there exists $\phi \in C_b^\infty \cap \mathcal{E}(\mathbb{R}^N)$ with $\nabla\phi \in H^\infty(\mathbb{R}^N)$, and $w \in H^1(\mathbb{R}^N)$ such that $u = \phi + w$.*

Lemma 5.5. *$\mathcal{E}(\mathbb{R}^N)$ is a complete metric space with the pseudometric distance*

$$d(v_1, v_2) = \|v_1' - v_2'\|_{L^2(\mathbb{R})} + \||v_1| - |v_2|\|_{L^2(\mathbb{R})},$$

$\mathcal{E}(\mathbb{R}^N) + H^1(\mathbb{R}^N) \subset \mathcal{E}(\mathbb{R}^N)$ and the maps

$$u \in \mathcal{E}(\mathbb{R}^N) \mapsto \nabla u \in L^2(\mathbb{R}^N),\ u \in \mathcal{E}(\mathbb{R}^N) \mapsto 1 - |u|^2 \in L^2(\mathbb{R}^N),$$
$$(u, w) \in \mathcal{E}(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \mapsto u + w \in \mathcal{E}(\mathbb{R}^N)$$

are continuous.

By using these lemmas we can conclude that

Theorem 5.6. *Assume that $\mathcal{W}$ satisfies the conditions in Theorem 5.1 or in Theorem 5.3. Then for every $\Psi_0 \in \mathcal{E}(\mathbb{R})$, there exists a unique $\Psi \in C(\mathbb{R}, \mathcal{E}(\mathbb{R}))$ global solution to* (NGP) *with the initial condition Ψ_0. Moreover, the energy is conserved*

5.3 Traveling waves

There have been extensive studies concerning the existence and stability of traveling waves in the case of the contact interaction $\mathcal{W} = \delta_0$, more commonly refereed to as dark solitons due to the nonzero boundary condition, (see [BS99, BGS09, BOS04, BGS15b, CM17, Mar13] and the references therein). However, there are very few mathematical results concerning general nonlocal interactions with nonzero conditions at infinity. More precisely, we focus now on solutions of the form

$$\Psi_c(x,t) = u(x-ct),$$

representing a traveling wave propagating at speed c. Hence, the profile u satisfies the nonlocal equation

$$ic\partial_{x_1}u + \Delta u + u(\mathcal{W} * (1-|u|^2)) = 0, \quad \text{in } \mathbb{R}^N. \tag{TW$_{\mathcal{W},c}$}$$

By taking the conjugate of the function, we assume without loss of generality that $c \geq 0$.

Let us remark that when considering vanishing boundary conditions at infinity, this kind of equation has been studied extensively [GV80, Caz03, MVS13] and long-range dipolar interactions in condensates have received recently much attention [LMS$^+$09, CMS08, AS11, BJ16, LS]. However, the techniques used in these works cannot be adapted to include solutions satisfying $|u(x)| \to 1$, as $|x| \to \infty$.

When $\mathcal{W}$ is given by a Dirac delta function, equation (TW$_{\delta_0,c}$) reduces to

$$ic\partial_{x_1}u + \Delta u + u(1-|u|^2) = 0, \quad \text{in } \mathbb{R}^N. \tag{5.10}$$

In dimension $N = 1$, this equation can be solved explicitly. As explained in [BGS08b], if $c \geq \sqrt{2}$, the only solutions in $\mathcal{E}(\mathbb{R})$ are the trivial ones (i.e. the constant functions of modulus one) and if $0 \leq c < \sqrt{2}$, the nontrivial solutions are given, up to invariances (translations and a multiplications by constants of modulus one), by

$$u_c(x) = \sqrt{\frac{2-c^2}{2}} \tanh\left(\frac{\sqrt{2-c^2}}{2}x\right) - i\frac{c}{\sqrt{2}}. \tag{5.11}$$

Thus, there is a family of dark solitons belonging to $\mathcal{NE}(\mathbb{R})$ for $c \in (0, \sqrt{2})$ and there is one stationary black soliton associated with the speed $c = 0$. Notice also that the values of $u_c(\infty)$ and $u_c(-\infty)$ are different.

In the higher dimension case, we have the following existence and nonexistence results.

Theorem 5.7 ([Mar13, BR])**.**

- *Let $N = 2$. For almost every $c \in (0, \sqrt{2})$, there is a nonconstant to* (5.10) *in $\mathcal{E}(\mathbb{R}^2)$.*
- *Let $N = 3$. For every $c \in (0, \sqrt{2})$, there is a nonconstant to* (5.10) *in $\mathcal{E}(\mathbb{R}^3)$.*

Theorem 5.8 ([BMR94, BS99, Gra03, Gra04])**.** *Let $v \in \mathcal{E}(\mathbb{R}^N)$ be a solution of* (5.10)*. Assume that one of the following cases hold*

(i) *$c = 0$.*

(ii) *$c > \sqrt{2}$.*

(iii) *$N = 2$ and $c = \sqrt{2}$.*

Then v is a constant function of modulus one.

It would be reasonable to expect to generalize in some way these theorems to the nonlocal equation ($\mathrm{TW}_{\mathcal{W},c}$). The aim of the papers [dL12] and [dLM20] was to investigate existence and nonexistence results depending on $\mathcal{W}$. Before explaining our results, we give some motivation about the critical speed. Let us proceed formally and consider a constant function u_0 of modulus one. Since (NGP) is invariant by a change of phase, we can assume $u_0 = 1$. Then the linearized equation of (NGP) at u_0 is given by

$$i\partial_t \tilde{u} - \Delta\tilde{u} + 2W * \mathrm{Re}(\tilde{u}) = 0. \tag{5.12}$$

Writing $\tilde{u} = \tilde{u}_1 + i\tilde{u}_2$ and taking real and imaginary parts in (5.12), we get

$$\begin{aligned} -\partial_t \tilde{u}_2 - \Delta\tilde{u}_1 + 2\mathcal{W} * \tilde{u}_1 &= 0, \\ \partial_t \tilde{u}_1 - \Delta\tilde{u}_2 &= 0, \end{aligned}$$

from where we deduce that

$$\partial^2_{tt}\tilde{u} - 2\mathcal{W} * (\Delta\tilde{u}) + \Delta^2\tilde{u} = 0. \tag{5.13}$$

By imposing $\tilde{u} = e^{i(\xi\cdot x - wt)}$, $w \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, as a solution of (5.13), we obtain the dispersion relation

$$w(\xi) = \pm\sqrt{|\xi|^4 + 2\widehat{\mathcal{W}}(\xi)|\xi|^2}, \tag{5.14}$$

Supposing that $\widehat{\mathcal{W}}$ is positive and continuous at the origin, we get in the long wave regime, i.e. $\xi \sim 0$,

$$w(\xi) \sim (2\widehat{\mathcal{W}}(0))^{1/2}|\xi|.$$

Consequently, in this regime we can identify $(2\widehat{\mathcal{W}}(0))^{1/2}$ as the speed of sound waves (also called sonic speed). In order to simplify our computations, we can normalize the equation so that the critical speed is fixed. Indeed, rescaling the equation, we can replace $\widehat{\mathcal{W}}(\xi)$ by $\widehat{\mathcal{W}}(\xi)/\widehat{\mathcal{W}}(0)$ in (NGP). Therefore, we assume from now on that $\widehat{\mathcal{W}}(0) = 1$ and hence that the critical speed is

$$c_* = \sqrt{2}.$$

Let us mention that the dispersion relation (5.14) was first observed by Bogoliubov [Bog47] on the study of Bose–Einstein gas. Under some physical considerations, he argued that the gas should move with a speed less than c_* to preserve its superfluid properties. In terms of equation ($\mathrm{TW}_{\mathcal{W},c}$), this leads to the conjecture that there are no nontrivial solution if $c > \sqrt{2}$.

For the sake of simplicity, we explain our results assuming from now on that $\mathcal{W} \in \mathcal{M}_2(\mathbb{R}^N) \cap \mathcal{M}_3(\mathbb{R}^N)$ *is a real-valued even tempered distribution, with* $\widehat{\mathcal{W}}$ *is of class* $C^3_b(\mathbb{R})$, i.e. of class C^3 whose first 3 derivatives are bounded.

Theorem 5.9 ([dL12]). *Let* $2 \le N \le 3$. *Assume that the map* $\xi \to \xi_j\partial_k\widehat{\mathcal{W}}(\xi)$ *is bounded on* $\mathbb{R}^N$, *for all* $j,k \in \{1,\dots,N\}$, *and that*

$$\widehat{\mathcal{W}}(\xi) \ge \max\left\{1, \frac{2}{N-1}\right\}\sum_{k=2}^N |\xi_k\partial_k\widehat{\mathcal{W}}(\xi)| + |\xi_1\partial_1\widehat{\mathcal{W}}(\xi)|, \quad \text{for all } \xi \in \mathbb{R}^N.$$

If $c > \sqrt{2}$, *then there is no nonconstant solution to* ($\mathrm{TW}_{\mathcal{W},c}$).

Concerning the static waves, we have the following result.

Theorem 5.10 ([dL12]). *Let $2 \le N \le 3$. Assume that the map $\xi \to \xi_j \partial_k \widehat{\mathcal{W}}(\xi)$ is bounded on $\mathbb{R}^N$, for all $j,k \in \{1,\dots,N\}$, and that*

$$\xi_j \partial_j \widehat{\mathcal{W}}(\xi) \le 0, \qquad \textit{for all } \xi \in \mathbb{R}^N,$$

for all $j \in \{1,\dots,N\}$. Then there is no nonconstant solution to (TW$_{\mathcal{W},c}$) *for $c = 0$.*

Let us give some examples of application of Theorem 5.9.

(a) In the case $\mathcal{W} = \delta_0$, we have $\nabla\widehat{\mathcal{W}} = 0$. Thus these theorems recover Theorem 5.8 in the cases (i) and (ii), i.e. nonexistence for all $c \in \{0\} \cup (\sqrt{2}, \infty)$.

(b) Consider the potential

$$\mathcal{W}_\varepsilon = \frac{1}{a}(\delta + \varepsilon f), \quad \varepsilon \ge 0,$$

where f is an even real-valued function, such that $f, |x|^2 f, |x|\nabla f \in L^1(\mathbb{R}^N)$, and $a = 1 + \varepsilon \widehat{f}(0)$. Then, we can compute $\varepsilon_0 > 0$ in term of f, such that for all $\varepsilon \in [0, \varepsilon_0)$ there is nonexistence of nontrivial solutions to (TW$_{\mathcal{W},c}$) in $\mathcal{E}(\mathbb{R}^N)$, for any $c \in (\sqrt{2}, \infty)$, in dimensions $2 \le N \le 3$.

(c) For the potential

$$\widehat{\mathcal{W}}(\xi) = \frac{1}{(1 + a|\xi|^2)^{b/2}},$$

we deduce nonexistence of nontrivial solutions of (TW$_{\mathcal{W},c}$) in $\mathcal{E}(\mathbb{R}^N)$, in the following cases:

(1) $N = 2$, $b \le 1/2$, $c \in (c_s, \infty)$.
(2) $N = 2$, $b > 1/2$, $c \in (c_s, \sqrt{2 + 2/b})$.
(3) $N = 3$, $b \le 1$, $c \in (c_s, \infty)$.
(4) $N = 3$, $b > 1$, $c \in (c_s, \sqrt{2 + 2/b})$.
(5) $N = 2$ or 3, $c = 0$.

We recall that Theorem 5.8-(i) follows from a Pohozaev identity. Gravejat in [Gra03] proved Theorem 5.8-(ii) by combining the respective Pohozaev identity with an integral equality obtained from the Fourier analysis of the equation satisfied by the function $\eta := 1 - |u|^2$. Our results are derived in the same spirit.

Let us suppose that $u \in \mathcal{E}(\mathbb{R}^N)$ is a solution to (TW$_{\mathcal{W},c}$) with speed c. The first step is to apply the elliptic regularity theory to (TW$_{\mathcal{W},c}$), in order to show that u is smooth. Moreover, $\eta := 1 - |u|^2$ and ∇u belong to $\mathcal{W}^{k,p}(\mathbb{R}^N)$, for all $k \in \mathbb{N}$, $2 \le p \le \infty$, and

$$|u(x)| \to 1, \ \nabla u(x) \to 0, \qquad \text{as } |x| \to \infty.$$

Furthermore, there exists a smooth lifting of u. More precisely, there exist $R_0 > 0$ and a smooth real-valued function θ defined on $B(0,R_0)^c$, with $\nabla\theta \in W^{k,p}(B(0,R_0)^c)$, for all $k \in \mathbb{N}$, $2 \le p \le \infty$, such that

$$\rho \ge \frac{1}{2} \quad \text{and} \quad u = \rho e^{i\theta} \quad \text{on } B(0,R_0)^c.$$

The most technical part of the proof is to establish rigorously that the following Pohozaev identities hold

$$
\begin{aligned}
E(u) &= \int_{\mathbb{R}^N} |\partial_1 u|^2 + \frac{1}{4(2\pi)^N} \int_{\mathbb{R}^N} \xi_1 \partial_1 \widehat{W} |\widehat{\eta}|^2 \, d\xi, \\
E(u) &= \int_{\mathbb{R}^N} |\partial_j u|^2 - \frac{c}{2} \int_{\mathbb{R}^N} (u_1 \partial_1 u_2 - u_2 \partial_1 u_1 - \partial_1(\chi\theta)) + \frac{1}{4(2\pi)^N} \int_{\mathbb{R}^N} \xi_j \partial_j \widehat{W} |\widehat{\eta}|^2 \, d\xi,
\end{aligned}
$$

for all $j \in \{2, \dots, N\}$. Here the function $\chi \in C^\infty(\mathbb{R}^N)$ is a cut-off function satisfying $|\chi| \leq 1$, $\chi = 0$ on $B(0, 2R_0)$ and $\chi = 1$ on $B(0, 3R_0)^c$. Indeed, to prove these formulas by using classical arguments, we would need that $x_j \eta \in L^2(\mathbb{R}^N)$. We handle this difficulty by carefully analyzing the integrals involved in the Fourier variable.

On the other hand, using ($\mathrm{TW}_{\mathcal{W},c}$), we obtain the equation for η,

$$
\Delta^2 \eta - 2\Delta(W * \eta) + c^2 \partial_{11}^2 \eta = -\Delta F + 2c\partial_1(\operatorname{div} G), \quad \text{ in } \mathbb{R}^N.
$$

where $G := v_1 \nabla v_2 - v_2 \nabla v_1 - \nabla(\chi\theta)$ and $F := 2|\nabla v|^2 + 2\eta(W * \eta) + 2cG_1$. This equation allows us to get the following integral identities, for all $j \in \{2, \dots, N\}$,

$$
\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta(W * \eta)) = -c \frac{\ell_{j,c}}{1 + \ell_{j,c}} \int_{\mathbb{R}^N} (v_1 \partial_1 v_2 - v_2 \partial_1 v_1 - \partial_1(\chi\theta)), \tag{5.15}
$$

where $\ell_{j,c}$ are constants depending on $\widehat{\mathcal{W}}$ and c. Finally, by combining the Pohozaev identities and (5.15), we can write an algebraic system for the quantities involved (kinetic and potential energy, momentum and nonlocal remainder terms), and then invoke the Farkas' lemma to obtain the desired nonexistence results.

5.4 Existence of traveling waves in dimension one

Concerning the existence of solutions to ($\mathrm{TW}_{\mathcal{W},c}$), it does not seem possible to find explicit solution to ($\mathrm{TW}_{\mathcal{W},c}$), in the presence of a nonlocal interaction $\mathcal{W}$. In the article [dLM20], we have found sufficient conditions on $\mathcal{W}$ in order to prove the existence of a branch of solutions, by using a variational approach.

Before going any further, let us state the assumptions that we need on $\mathcal{W}$.

(H1) $\mathcal{W}$ is an even tempered distribution with $\widehat{\mathcal{W}} \in C_b^3(\mathbb{R})$, and $\widehat{\mathcal{W}}(\xi) \geq (1 - \xi^2/2)^+$, for all $\xi \in \mathbb{R}$. Moreover,

$$
\widehat{\mathcal{W}}(0) = 1 \quad \text{and} \quad (\widehat{\mathcal{W}})''(0) > -1.
$$

(H2) $\widehat{\mathcal{W}}$ admits a meromorphic extension to the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, and the only possible singularities of $\widehat{\mathcal{W}}$ on $\mathbb{H}$ are simple isolated poles belonging to the imaginary axis, i.e. they are given by $\{i\nu_j : j \in J\}$, with $\nu_j > 0$, for all $j \in J$, $0 \leq \operatorname{Card} J \leq \infty$, and their residues $\operatorname{Res}(\widehat{\mathcal{W}}, i\nu_j)$ are purely imaginary numbers satisfying

$$
i \operatorname{Res}(\widehat{\mathcal{W}}, i\nu_j) \leq 0, \quad \text{for all } j \in J,
$$

Also, there exists a sequence of rectifiable curves $(\Gamma_k)_{k\in\mathbb{N}^*} \subset \mathbb{H}$, parametrized by $\gamma_k : [a_k, b_k] \to \mathbb{C}$, such that $\Gamma_k \cup [-k, k]$ is a closed positively oriented simple curve that does not pass through any poles. Moreover,

$$\lim_{k\to\infty} |\gamma_k(t)| = \infty, \text{ for all } t \in [a_k, b_k], \quad \text{and} \quad \lim_{k\to\infty} \operatorname{length}(\Gamma_k) \sup_{t\in[a_k,b_k]} \frac{\widehat{\mathcal{W}}(\gamma_k(t))}{|\gamma_k(t)|^4} = 0. \tag{5.16}$$

Let us remark that since $\hat{\delta}_0 = 1$, the assumptions (H1)–(H2) are trivially fulfilled in the case $\mathcal{W} = \delta_0$. Let us make some further remarks about these hypotheses. Assumption (H1) ensures that the critical speed exists and that the energy functional is nonnegative and well defined in $\mathcal{E}(\mathbb{R})$.

As explained before, the condition $\widehat{\mathcal{W}}(0) = 1$ is just a choice of normalization. The condition $\widehat{\mathcal{W}}(\xi) \geq (1 - \xi^2/2)^+$ in hypothesis (H1) can be seen as a coercivity property for the energy. In particular, it allows us to use the key energy estimates in Lemma 5.2. The condition $(\widehat{\mathcal{W}})''(0) > -1$ is crucial to show that the behavior of a solution of $(\mathrm{TW}_{\mathcal{W},c})$ can be formally described in terms of the solution of the Korteweg–de Vries equation

$$(1 + (\widehat{\mathcal{W}})''(0))A'' - 6A^2 - A = 0, \tag{5.17}$$

at least for c close to $\sqrt{2}$.

The more technical and restrictive assumption (H2) is used only to prove that the curve associated with the minimizing problem is concave. Indeed, we use some ideas introduced by Lopes and Mariş [LM08] to study the minimization of the nonlocal functional

$$\int_{\mathbb{R}^N} m(\xi)|\hat{w}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} F(w(x))dx,$$

under the constraint $\int_{\mathbb{R}^N} G(w)dx = \lambda$, $\lambda \in \mathbb{R}$, for a class of symbols m (see (2.16) in [LM08]). Here $N \geq 2$, F and G are local functions, and the minimization is over $w \in H^s(\mathbb{R})$. The results in [LM08] cannot be applied to the symbol $m(\xi) = \widehat{\mathcal{W}}(\xi)$ nor to the minimization over functions with nonvanishing conditions at infinity (nor $N = 1$). However, we can still apply the reflexion argument in [LM08], which will lead us to show that

$$\int_{\mathbb{R}} (\mathcal{W} * f)f \geq \int_{\mathbb{R}} (\mathcal{W} * \tilde{f})\tilde{f}, \tag{5.18}$$

for all odd functions $f \in C_c^\infty(\mathbb{R})$, where $\tilde{f}$ is given by $\tilde{f}(x) = f(x)$ for $x \in \mathbb{R}^+$, and $\tilde{f}(x) = -f(x)$ for $x \in \mathbb{R}^-$. Using the sine and cosine transforms

$$\hat{f}_s(\xi) = \int_0^\infty \sin(x\xi)f(x)dx, \quad \hat{f}_c(\xi) = \int_0^\infty \cos(x\xi)f(x)dx,$$

inequality (5.18) is equivalent to the condition

$$\int_0^\infty \widehat{\mathcal{W}}(\xi)(|\hat{f}_s(\xi)|^2 - |\hat{f}_c(\xi)|^2)d\xi \geq 0, \tag{5.19}$$

for all odd functions $f \in C_c^\infty(\mathbb{R})$. By using Cauchy's residue theorem, it can be verified that assumption (H2) implies that inequality (5.19) holds.

5.4.1 The variational approach

It can be seen, at least formally, that

$$icu' + u'' + u(\mathcal{W} * (1 - |u|^2)) = 0, \quad \text{in } \mathbb{R}. \tag{TW1d$_{\mathcal{W},c}$}$$

corresponds to the Euler–Lagrange equation associated with the problem of minimizing the energy, under the constraint of fixed momentum

$$p(v) = \frac{1}{2}\int_{\mathbb{R}} \langle iv', v\rangle \left(1 - \frac{1}{|v|^2}\right),$$

that is well-defined in the nonvanishing energy space

$$\mathcal{NE}(\mathbb{R}) = \{v \in \mathcal{E}(\mathbb{R}) : \inf_{\mathbb{R}} |v| > 0\}.$$

In this manner, the speed c appears as a Lagrange multiplier.

Let us now describe our minimization approach for the existence problem. For $\mathfrak{q} \geq 0$, we consider the minimization curve

$$E_{\min}(\mathfrak{q}) := \inf\{E(v) : v \in \mathcal{NE}(\mathbb{R}),\ p(v) = \mathfrak{q}\}.$$

We can show that this curve is well-defined and that is nondecreasing. We also set

$$\mathfrak{q}_* = \sup\{\mathfrak{q} > 0 \mid \forall v \in \mathcal{E}(\mathbb{R}), E(v) \leq E_{\min}(\mathfrak{q}) \Rightarrow \inf_{\mathbb{R}}|v| > 0\}.$$

If (H2) is also fulfilled and $\mathfrak{q} \in (0, \mathfrak{q}^*)$, we showed that minimum associated with $E_{\min}(\mathfrak{q})$ is attained and that the corresponding Euler–Lagrange equation satisfied by the minimizers is exactly (TW1d$_{\mathcal{W},c}$). More precisely, we have the existence of a family of solutions of (TW1d$_{\mathcal{W},c}$) parametrized by the momentum.

Theorem 5.11 ([dLM20]). *Assume that* (H1) *and* (H2) *hold. Then* $\mathfrak{q}_* > 0.027$ *and for all* $\mathfrak{q} \in (0, \mathfrak{q}_*)$ *there is a nontrivial solution* $u \in \mathcal{NE}(\mathbb{R})$ *to* (TW1d$_{\mathcal{W},c}$) *satisfying* $p(u) = \mathfrak{q}$*, for some* $c \in (0, \sqrt{2})$.

It is important to remark that the constant $\mathfrak{q}_*$ is not necessarily small. For instance, in the case $\mathcal{W} = \delta_0$, the explicit solution (5.11) allows us to compute the momentum of u_c, for $c \in (0, \sqrt{2})$, and to deduce that $\mathfrak{q}_* = \pi/2$. Moreover $E_{\min}$ can be determined and its profile is depicted in Figure 5.1. Notice that $E_{\min}$ is constant on $(\mathfrak{q}_*, \infty)$ and that in this interval the minimum is not attained (see e.g. [BGS08b]). Since (H1)–(H2) are satisfied by $\mathcal{W} = \delta_0$, and since there is uniqueness (up to invariances) of the solutions to (TW$_{\delta_0,c}$), we deduce that the branch of solutions given by Theorem 5.11 corresponds to the dark solitons in (5.11), for $c \in (0, \sqrt{2})$. In the general case, we do not know if the solution given by Theorem 5.11 is unique (up to invariances). Actually, the uniqueness for nonlocal equations such as (TW1d$_{\mathcal{W},c}$) can be difficult to establish (see e.g. [Alb95, Lie77]) and it would be an interesting subject to investigate.

To establish Theorem 5.11, we analyze two problems. First, we provide some general properties of the curve $E_{\min}$. Then, we study the compactness of the minimizing sequences associated with $E_{\min}$. The next result summarizes the properties of $E_{\min}$.

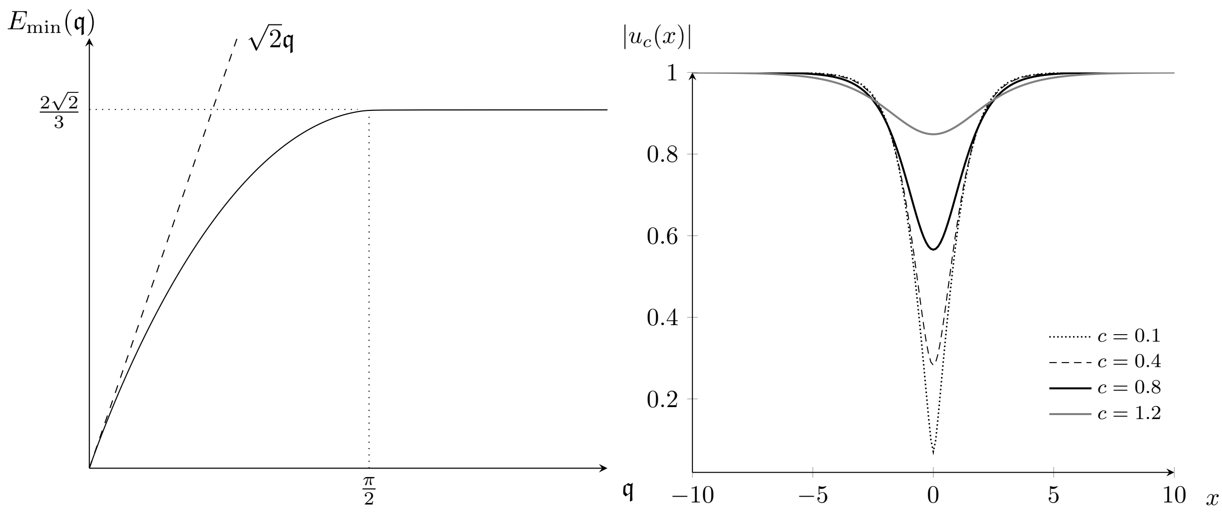


Figure 5.1: Curve $E_{\min}$ and solitons in the case $\mathcal{W} = \delta_0$.

Theorem 5.12 ([dLM20]). *Suppose that* $\mathcal{W}$ *satisfies* (H1). *Then the following statements hold.*

(i) *The function* $E_{\min}$ *is even and Lipschitz continuous on* $\mathbb{R}$, *with*

$$|E_{\min}(\mathfrak{p}) - E_{\min}(\mathfrak{q})| \leq \sqrt{2}|\mathfrak{p} - \mathfrak{q}|, \quad \text{for all } \mathfrak{p}, \mathfrak{q} \in \mathbb{R}.$$

Moreover, it is nondecreasing and subadditive on $\mathbb{R}^+$.

(ii) *There exist constants* $\mathfrak{q}_1, A_1, A_2, A_3 > 0$ *such that*

$$\sqrt{2}\mathfrak{q} - A_1\mathfrak{q}^{3/2} \leq E_{\min}(\mathfrak{q}) \leq \sqrt{2}\mathfrak{q} - A_2\mathfrak{q}^{5/3} + A_3\mathfrak{q}^2, \quad \text{for all } \mathfrak{q} \in [0, \mathfrak{q}_1].$$

(iii) *If* (H2) *is satisfied, then* $E_{\min}$ *is concave on* $\mathbb{R}^+$.

(iv) *We have* $\mathfrak{q}_* > 0.027$. *If* $E_{\min}$ *is concave on* $\mathbb{R}^+$, *then* $E_{\min}$ *is strictly increasing on* $[0, \mathfrak{q}_*)$, *and for all* $v \in \mathcal{E}(\mathbb{R})$ *satisfying* $E(v) < E_{\min}(\mathfrak{q}_*)$, *we have* $v \in \mathcal{NE}(\mathbb{R})$.

(v) *Assume that* $E_{\min}$ *is concave on* $\mathbb{R}^+$. *Then* $E_{\min}(\mathfrak{q}) < \sqrt{2}\mathfrak{q}$, *for all* $\mathfrak{q} > 0$, $E_{\min}$ *is strictly subadditive on* $\mathbb{R}^+$, *and the right and left derivatives of* $E_{\min}$, *denoted by* $E^+_{\min}$ *and* $E^-_{\min}$ *respectively, satisfy*

$$0 \leq E^+_{\min}(\mathfrak{q}) \leq E^-_{\min}(\mathfrak{q}) < \sqrt{2}.$$

Furthermore, $E^+_{\min}(\mathfrak{q}) \to E^+_{\min}(0) = \sqrt{2}$, *as* $\mathfrak{q} \to 0^+$.

To prove the existence of solutions we use a concentration-compactness argument. Applying Theorem 5.12, we show that the minimum is attained at least for $\mathfrak{q} \in (0, \mathfrak{q}_*)$, so that the set

$$\mathcal{S}_{\mathfrak{q}} = \{v \in \mathcal{NE}(\mathbb{R}) : E(v) = E_{\min}(\mathfrak{q}) \text{ and } p(v) = \mathfrak{q}\}$$

is nonempty, and thus there are nontrivial solutions to (TW1d$_{\mathcal{W},c}$). Hence, we can rely on the Cazenave–Lions [CL82] argument to show that the solutions are orbitally stable. Precisely, we say that the set $\mathcal{S}_{\mathfrak{q}}$ is orbitally stable in $(\mathcal{E}(\mathbb{R}),d)$ if for all $\Psi_0 \in \mathcal{E}(\mathbb{R})$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$d(\Psi_0, \mathcal{S}_{\mathfrak{q}}) \leq \delta,$$

then the solution $\Psi(t)$ of (NGP) associated with the initial condition Ψ_0 satisfies

$$\sup_{t\in\mathbb{R}} d(\Psi(t), \mathcal{S}_{\mathfrak{q}}) \leq \varepsilon.$$

Now we can state our main result concerning the existence and stability of traveling waves.

Theorem 5.13 ([dLM20]). *Suppose that $\mathcal{W}$ satisfies* (H1) *and that $E_{\min}$ is concave on $\mathbb{R}^+$. Then the set $\mathcal{S}_{\mathfrak{q}}$ is nonempty, for all $\mathfrak{q} \in (0, \mathfrak{q}_*)$. Moreover, every $u \in \mathcal{S}_{\mathfrak{q}}$ is a solution of* (TW1d$_{\mathcal{W},c}$) *for some speed $c_{\mathfrak{q}} \in (0, \sqrt{2})$ satisfying $E^+_{\min}(\mathfrak{q}) \leq c_{\mathfrak{q}} \leq E^-_{\min}(\mathfrak{q})$. Also, $c_{\mathfrak{q}} \to \sqrt{2}$ as $\mathfrak{q} \to 0^+$.*

Furthermore, if $\mathcal{W} \in \mathcal{M}_3(\mathbb{R})$, then $\mathcal{S}_{\mathfrak{q}}$ is orbitally stable in $(\mathcal{E}(\mathbb{R}), d)$.

In this manner, it is clear that Theorem 5.11 is an immediate corollary of Theorems 5.12 and 5.13, and that the branch of solutions given by Theorem 5.11 is orbitally stable. In particular, we recover the orbital stability proved by several authors for the solitons given in (5.11) (see e.g. [Lin02, BS99, Chi13] and the references therein).

We point out that we have not discussed what happens with the minimizing curve for $\mathfrak{q} \geq \mathfrak{q}_*$. As mentioned before, for all $\mathfrak{q} > \mathfrak{q}_*$, the curve $E_{\min}(\mathfrak{q})$ is constant for $\mathcal{W} = \delta_0$ (see Figure 5.1) and $\mathcal{S}_q$ is empty. Moreover, the critical case $\mathfrak{q} = \mathfrak{q}_*$ is associated with the black soliton and its analysis is more involved (see e.g. [BGSS08b, GS15b]). Numerical simulations lead us to conjecture that similar results hold for a potential satisfying (H1)-(H2), i.e. that $E_{\min}(\mathfrak{q})$ is constant and that $\mathcal{S}_q$ is empty on $(\mathfrak{q}_*, \infty)$, and that there is a black soliton when $\mathfrak{q} = \mathfrak{q}_*$. In addition, in the performed simulations the value $\mathfrak{q}_*$ is close to $\pi/2$ (see Subsection 5.4.3). Furthermore, these simulations also show that (H2) is not necessary for the concavity of $E_{\min}$ nor the existence of solutions of (TW1d$_{\mathcal{W},c}$). As seen from Theorem 5.12, we have only used (H2) as a sufficient condition to ensure the concavity of $E_{\min}$. If for some $\mathcal{W}$ satisfying (H1), one is capable of showing that $E_{\min}$ is concave, then the existence and stability of solutions of (TW1d$_{\mathcal{W},c}$) is a consequence of Theorem 5.13.

In addition to the smoothness of the obtained solutions, it is possible to study further properties of these solitons such as their decay at infinity and uniqueness (up to invariances). Another related open problem is to show the nonexistence of traveling waves for $c > \sqrt{2}$. These are open questions that could lead to further works.

We end this subsection by giving some examples of potentials satisfying conditions (H1)–(H2)

(i) For $\beta > 2\alpha > 0$, we consider $\mathcal{W}_{\alpha,\beta} = \frac{\beta}{\beta-2\alpha}(\delta_0 - \alpha e^{-\beta|x|})$, so its Fourier transform is

$$\widehat{\mathcal{W}}_{\alpha,\beta}(\xi) = \frac{\beta}{\beta - 2\alpha}\Big(1 - \frac{2\alpha\beta}{\xi^2 + \beta^2}\Big),$$

so that $\widehat{\mathcal{W}}_{\alpha,\beta}(0) = 1$, and it is simple to check that (H1) is satisfied. To verify (H2), it is enough to notice that the only singularity on $\mathbb{H}$ of the meromorphic function $\widehat{\mathcal{W}}_{\alpha,\beta}$ is the simple pole $\nu_1 = i\beta$ and that

$$i\,\mathrm{Res}(\widehat{\mathcal{W}}_{\alpha,\beta}, i\beta) = -\frac{\alpha\beta}{\beta - 2\alpha} < 0.$$

Since $\widehat{\mathcal{W}}_{\alpha,\beta}$ is bounded on $\mathbb{H}$ away from the pole, we conclude that (H2) is fulfilled. We recall that, by the Young inequality, $L^1(\mathbb{R})$ is a subset of $\mathcal{M}_3(\mathbb{R})$. Therefore $\mathcal{W}_{\alpha,\beta} \in \mathcal{M}_3(\mathbb{R})$ and Theorem 5.13 applies.

(ii) For $\alpha \in [0,1)$, we take the potential $\mathcal{W}_\alpha = \frac{1}{1-\alpha}(\delta_0 - \alpha\mathcal{V})$, where

$$\mathcal{V}(x) = -\frac{3}{\pi}\ln(1 - e^{-\pi|x|}), \quad \text{and} \quad \widehat{\mathcal{V}}(\xi) = \frac{3(\xi\coth(\xi) - 1)}{\xi^2}.$$

It can be seen that $\widehat{\mathcal{V}}$ is a smooth even positive function on $\mathbb{R}$, decreasing on $\mathbb{R}^+$, with $\widehat{\mathcal{V}}(0) = 1$ and decaying at infinity as $3/\xi$. Thus, condition (H1) is satisfied. As a function on the complex plane, $\widehat{\mathcal{V}}$ is a meromorphic function whose only singularities on $\mathbb{H}$ are given by the simple poles $\{i\pi\ell\}_{\ell\in\mathbb{N}^*}$, and

$$i\,\mathrm{Res}(\widehat{\mathcal{W}}_\alpha, i\pi\ell) = i\,\mathrm{Res}(-\widehat{\mathcal{V}}, i\pi\ell) = -\frac{3}{\pi\ell}.$$

To check (H2), we define for $k \geq 2$, the functions $\gamma_{1,k}(t) = (k+1/2)\pi + it$, $t \in [0,(k+1/2)\pi]$, $\gamma_{2,k}(t) = t + i(k+1/2)\pi$, $t \in [(k+1/2)\pi, -(k+1/2)\pi]$, and $\gamma_{3,k}(t) = -(k+1/2)\pi + it$, $t \in [(k+1/2)\pi, 0]$, so that the corresponding curve Γ_k is given by the three sides of a square and Γ_k does not pass through any poles. Using that for $x, y \in \mathbb{R}$ (see e.g. [AS64])

$$|\coth(x+iy)| = \Big|\frac{\cosh(2x) + \cos(2y)}{\cosh(2x) - \cos(2y)}\Big|^{1/2},$$

we can obtain a constant $C > 0$, independent of k, such that $|\widehat{\mathcal{V}}(\gamma_{j,k}(t))| \leq C$, for all $t \in [a_{j,k}, b_{j,k}]$, for $j \in \{1,2,3\}$, where $[a_{j,k}, b_{j,k}]$ is the domain of definition of $\gamma_{j,k}$. As a conclusion, (H2) is fulfilled. Since $V \in L^1(\mathbb{R})$, we conclude that $\mathcal{W}_\alpha \in \mathcal{M}_3(\mathbb{R})$ and therefore we can apply Theorem 5.13 to this potential.

(iii) We can also construct perturbations of previous examples. For instance, using the function $\mathcal{V}$ defined above, we set

$$\widehat{\mathcal{W}}_{\sigma,m}(\xi) = \frac{2m^2\pi^2}{m^2\pi^2 + 2\sigma}\Big(1 - \frac{\widehat{\mathcal{V}}(\xi)}{2} + \frac{\sigma}{\xi^2 + m^2\pi^2}\Big),$$

for $\sigma \in \mathbb{R}$ and $m \in \mathbb{N}^*$, so that the poles on $\mathbb{H}$ are still $i\pi\mathbb{N}^*$. It follows that for $\sigma > -\pi^2 m^2/2$, the potential satisfies (H1) for $\sigma \in (-\pi^2 m^2/2, 3]$, and that (H2) holds if $\sigma \leq 3$. Therefore Theorem 5.13 applies.

5.4.2 Ideas of the proofs

To study the minimizing sequences associated with the curve $E_{\min}$, we use a concentration-compactness argument. A key point to obtain the compactness of these sequences is that the momentum can be controlled by the energy. This kind of inequality is crucial in the arguments when proving the existence of solitons by variational techniques in the case $\mathcal{W} = \delta_0$ (see [BGS09, CM17]). More precisely, for an open set $\Omega \subset \mathbb{R}$ and $u = \rho e^{i\theta} \in \mathcal{NE}(\mathbb{R})$, we need to be able to control the localized momentum

$$p_\Omega(u) := \frac{1}{2}\int_\Omega \eta\theta',$$

by some localized version of the energy. By the Cauchy inequality, setting as usual $\eta = 1 - |u|^2$, we have

$$\sqrt{2}|p_\Omega(u)| \leq \frac{1}{4}\int_\Omega \eta^2 + \frac{1}{2}\int_\Omega \theta'^2 \leq \frac{1}{4}\int_\Omega \eta^2 + \frac{1}{2\inf\limits_\Omega \rho^2}\int_\Omega \rho^2\theta'^2,$$

but it is not clear how to define a localized version of energy, due the to the nonlocal interactions. We propose to introduce the localized energy

$$E_\Omega(u) := \frac{1}{2}\int_\Omega |u'|^2 + \frac{1}{4}\int_{\mathbb{R}} (\mathcal{W} * \eta_\Omega)\eta_\Omega, \qquad \text{with } \eta_\Omega := \eta \mathbb{1}_\Omega.$$

Notice that if $\Omega = \mathbb{R}$, then $E_\Omega(u) = E(u)$ and $p_\Omega(u) = p(u)$. Since η_Ω can be discontinuous (and thus not weakly differentiable) when Ω is bounded, we also need to introduce a smooth cut-off function as follows: for Ω_0 an open set compactly contained in Ω, i.e. $\Omega_0 \subset\subset \Omega$, we set a function $\chi_{\Omega,\Omega_0} \in C^\infty(\mathbb{R})$ taking values in $[0,1]$ and satisfying $\chi_{\Omega,\Omega_0} \equiv 1$ on Ω_0 and $\chi_{\Omega,\Omega_0} \equiv 0$ on $\mathbb{R} \setminus \Omega$. In the case $\Omega = \Omega_0 = \mathbb{R}$, we simply set $\chi_{\Omega,\Omega_0} \equiv 1$.

Lemma 5.14. *Let $\Omega, \Omega_0 \subset \mathbb{R}$ be two smooth open sets with $\Omega_0 \subset\subset \Omega$ and let $\chi_{\Omega,\Omega_0} \in C^\infty(\mathbb{R})$ as above. Let $u \in \mathcal{E}(\mathbb{R})$ and assume that there is some $\varepsilon \in (0,1)$ such that $1 - \varepsilon \leq |u|^2 \leq 1 + \varepsilon$ on Ω. Then*

$$\sqrt{2}|p_\Omega(u)| \leq \frac{E_\Omega(u)}{1-\varepsilon} + \Delta_{\Omega,\Omega_0}(u),$$

where the remainder term $\Delta_{\Omega,\Omega_0}(u)$ satisfies the estimate

$$|\Delta_{\Omega,\Omega_0}(u)| \leq C(\|\eta\|_{L^2(\Omega\setminus\Omega_0)} + \|\eta\chi'_{\Omega,\Omega_0}\|_{L^2(\Omega\setminus\Omega_0)} + \|\eta\chi'_{\Omega,\Omega_0}\|^2_{L^2(\Omega\setminus\Omega_0)}).$$

Here $C = C(E(u), \varepsilon)$ is a constant depending on $E(u)$ and ε, but not on Ω nor Ω_0. In particular, in the case $\Omega = \Omega_0 = \mathbb{R}$, we have

$$|p(u)| \leq \frac{E(u)}{\sqrt{2}(1-\varepsilon)}.$$

Let us give the proof of Lemma 5.14 in the simplest case $\Omega = \mathbb{R}$ to explain how the condition $\widehat{\mathcal{W}}(\xi) \geq (1 - \xi^2/2)^+$ appears. In this case $\eta_\Omega = \eta$, so that using the Plancherel theorem, this assumption gives us for any $\sigma > 0$,

$$\frac{\sigma}{4}\int_{\mathbb{R}} \Big(\eta^2 - (\mathcal{W} * \eta)\eta\Big) = \frac{\sigma}{8\pi}\int_{\mathbb{R}} |\widehat{\eta}|^2\Big(1 - \widehat{\mathcal{W}}(\xi)\Big) \leq \frac{\sigma}{16\pi}\int_{\mathbb{R}} \xi^2|\widehat{\eta}|^2 = \frac{\sigma}{8}\int_{\mathbb{R}} (\eta')^2.$$

Therefore, using the Cauchy inequality and that $1-\varepsilon \leq \rho^2 \leq 1+\varepsilon$,

$$
\begin{aligned}
\sqrt{2}|p(u)| &\leq \frac{\sigma}{4}\int_{\mathbb{R}} \eta^2 + \frac{1}{2\sigma(1-\varepsilon)}\int_{\mathbb{R}} \rho^2\theta'^2 \\
&= \frac{\sigma}{4}\int_{\mathbb{R}} \Big(\eta^2 - (\mathcal{W}*\eta)\eta\Big) + \frac{\sigma}{4}\int_{\mathbb{R}}(\mathcal{W}*\eta)\eta + \frac{1}{2\sigma(1-\varepsilon)}\int_{\mathbb{R}}\rho^2\theta'^2 \\
&\leq \frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon}\int_{\Omega}\Big(\frac{\rho'^2}{2} + \frac{\rho^2\theta'^2}{2}\Big) + \frac{1}{4\sqrt{1-\varepsilon^2}}\int_{\mathbb{R}}(\mathcal{W}*\eta)\eta \leq \frac{E(u)}{\sqrt{2}(1-\varepsilon)},
\end{aligned}
$$

where we have used that $\eta'^2 \leq 4(1+\varepsilon)\rho'^2$ and taken $\sigma = 1/\sqrt{1-\varepsilon^2}$.

Concerning the properties of $E_{\min}(\mathfrak{q})$, it is simply to prove that $E_{\min}$ is well-defined on $\mathbb{R}$, that is a continuous function and that $E_{\min} \leq \sqrt{2}\mathfrak{q}$. Moreover, as a consequence of Lemma 5.14, we get the following estimates near the origin.

Proposition 5.15. *There are constants $\mathfrak{q}_0 > 0$ and $K_0 > 0$ such that*

$$
\sqrt{2}\mathfrak{q} - K_0\mathfrak{q}^{3/2} \leq E_{\min}(\mathfrak{q}), \quad \text{for all } \mathfrak{q} \in [0,\mathfrak{q}_0).
$$

We also need upper estimates for the curve, which is much more involved, and requires using the soliton KdV equation (5.17), namely

$$
A(x) := -\frac{1}{4}\operatorname{sech}^2\big(\frac{x}{2\omega}\big).
$$

In fact, we expect that the solitons we are looking for behave as $(1+\varepsilon^2 A(\varepsilon x))e^{i\varepsilon\varphi(\varepsilon x)}$, where $\varphi' = -\sqrt{2}A$.

Lemma 5.16. *Let $v_\varepsilon(x) = (1+\varepsilon^2 A(\varepsilon x))e^{i\varepsilon\varphi(\varepsilon x)}$. Then*

$$
E(v_\varepsilon) = \frac{\omega}{3}\left(\varepsilon^3 - \frac{\varepsilon^5}{4}\right) + \mathcal{O}(\varepsilon^6) \quad \text{and} \quad p(v_\varepsilon) = \frac{\sqrt{2}\omega}{6}\left(\varepsilon^3 - \frac{\varepsilon^5}{10}\right), \tag{5.20}
$$

where $\mathcal{O}(\varepsilon^7)/\varepsilon^7$ is a function that is bounded in terms of $\|\widehat{\mathcal{W}}\|_{W^{3,\infty}}$, uniformly for all $\varepsilon \in (0,1]$.

From (5.20), we obtain the upper bound for $E_{\min}(\mathfrak{q})$ in Theorem 5.12-(ii). In particular, assuming that $E_{\min}$ is concave, we deduce that $E_{\min}(\mathfrak{q}) < \sqrt{2}\mathfrak{q}$, for all $\mathfrak{q} > 0$. We then introduce the quantity

$$
\Sigma_{\mathfrak{q}} := 1 - \frac{E_{\min}(\mathfrak{q})}{\sqrt{2}\mathfrak{q}},
$$

that is strictly positive for $\mathfrak{q} > 0$. Setting

$$
X_{\mathfrak{q},\delta} := \{v \in \mathcal{NE}(\mathbb{R}) : |p(v) - \mathfrak{q}| \leq \delta \text{ and } |E(v) - E_{\min}(\mathfrak{q})| \leq \delta\},
$$

and invoking again Lemma 5.14, we thus infer that for all $L > 1$, there is $\delta_0 > 0$ such that for all $\delta \in [0,\delta_0]$ and for all $v \in X_{q,\delta}$, there exists $\bar{x} \in \mathbb{R}$ such that

$$
\big|1 - |v(\bar{x})|^2\big| \geq \frac{\Sigma_{\mathfrak{q}}}{L}.
$$

This property allows us to avoid the vanishing case of the function $\eta := 1 - |v|^2$, in the concentration-compactness argument. The dichotomy is handle by using the concavity of the curve $E_{\min}$.

5.4.3 Numerical simulations

In this subsection, we numerically illustrate the properties of the minimizing curve through some simulations.

First, we show our results for the examples (i) and (ii) in Section 5.1. In Figures 5.2 and 5.3, we can see $E_{\min}$ and the modulus of the solitons associated with $\mathfrak{q} = 0.05$, $\mathfrak{q} = 0.55$, $\mathfrak{q} = 1.1$ and $\mathfrak{q} = 1.5$, for the potentials

$$\mathcal{W}_{\alpha,\beta} = \frac{\beta}{\beta - 2\alpha}(\delta_0 - \alpha e^{-\beta|x|}), \tag{5.21}$$

with $\alpha = 0.05$, $\beta = 0.15$, and

$$\mathcal{W}_{\alpha} = \frac{1}{1-\alpha}\big(\delta_0 + \frac{3\alpha}{\pi}\ln(1 - e^{-\pi|x|})\big), \tag{5.22}$$

with $\alpha = 0.8$. In both cases, we observe that $E_{\min}$ is concave and that the line $\sqrt{2}\mathfrak{q}$ is a

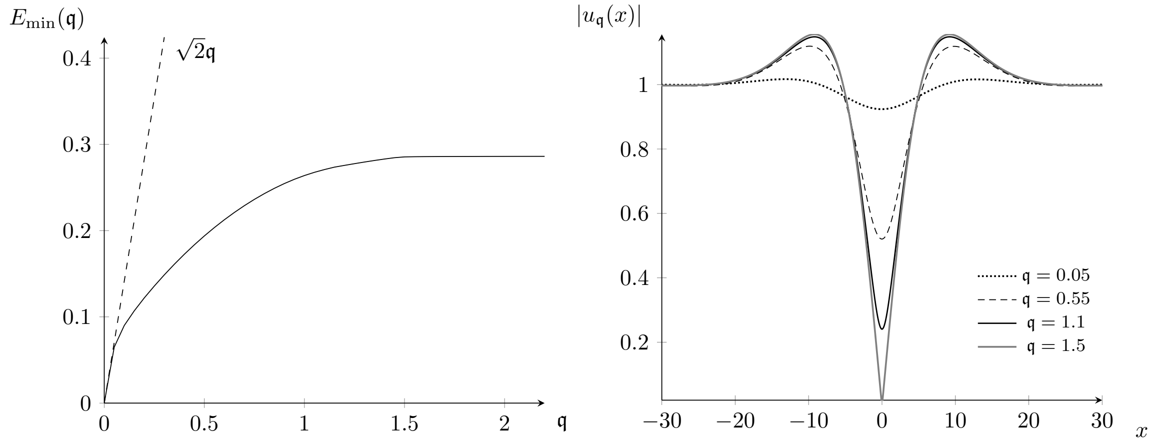


Figure 5.2: Curve $E_{\min}$ and solitons for the potential in (5.21), with $\alpha = 0.05$ and $\beta = 0.15$.

tangent to the curve. We notice that the shapes of the solitons in Figure 5.3 and the solitons in Figure 5.1 are quite similar. On the other hand, the solitons in Figure 5.2 are very different, they have values greater than 1 and exhibit a bump on $\mathbb{R}^+$. Notice also that the curves $E_{\min}$ for both potentials seem to be constant for $\mathfrak{q} > 1.55$.

We end this manuscript showing some numerical simulations for two interesting potentials. The first one has been proposed in [VFK14] as simple model for interactions in a Bose–Einstein condensate. It is given by a contact interaction δ_0 and two Dirac delta functions centered at $\pm\sigma$,

$$\mathcal{W}_{\sigma} = 2\delta_0 - \frac{1}{2}\left(\delta_{\sigma} + \delta_{-\sigma}\right). \tag{5.23}$$

Noticing that $\widehat{\mathcal{W}_{\sigma}}(\xi) = 2 - \cos(\sigma\xi)$, we see that for $\sigma > 0$, $\mathcal{W}_{\sigma}$ fulfills (H1), and that $\widehat{\mathcal{W}_{\sigma}}$ is analytic in $\mathbb{C}$, but is exponentially growing on $\mathbb{H}$. Thus, $\mathcal{W}_{\sigma}$ does not satisfy the assumption

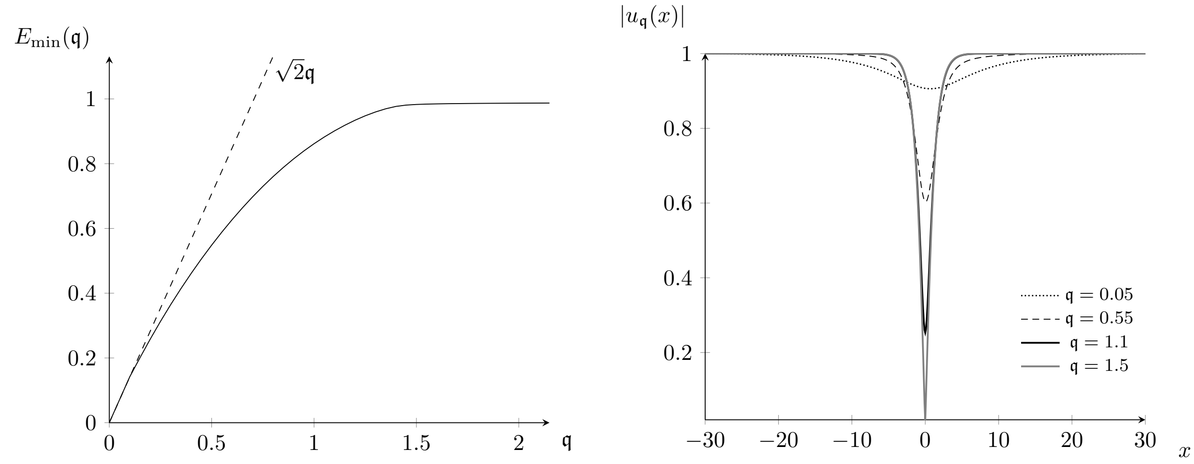


Figure 5.3: Curve $E_{\min}$ and solitons for the potential in (5.22), with $\alpha = 0.8$.

(5.16) in (H2). Nevertheless, the results of the simulation depicted in Figure 5.4 show that $E_{\min}$ is concave, and in that case Theorem 5.13 gives the orbital stability of the solitons illustrated in Figure 5.4.

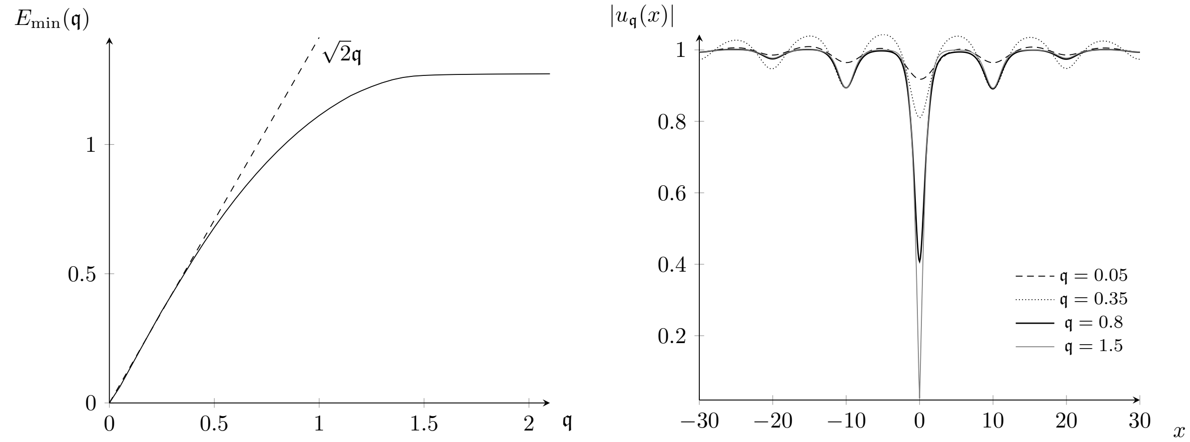


Figure 5.4: Curve $E_{\min}$ and solitons for the potential in (5.23), with $\sigma = 10$.

Finally, we consider the potential

$$\widehat{\mathcal{W}}_{a,b,c}(\xi) = (1 + a\xi^2 + b\xi^4)e^{-c\xi^2}, \tag{5.24}$$

that it has been proposed in [BR99, RSC18] to describe a quantum fluid exhibiting a roton-maxon spectrum such as Helium 4. Indeed, as predicted by the Landau theory, in such a fluid, the dispersion curve (5.14) cannot be monotone and it should have a local maximum and a local minimum, that are the so-called maxon and roton, respectively. In Figure 5.5, we see the dispersion curve associated with potential (5.24), with $a = -36$, $b = 2687$, $c = 30$. In this case, there is a maxon at $\xi_m \sim 0.33$ and a roton at $\xi_r \sim 0.53$. For these values, neither

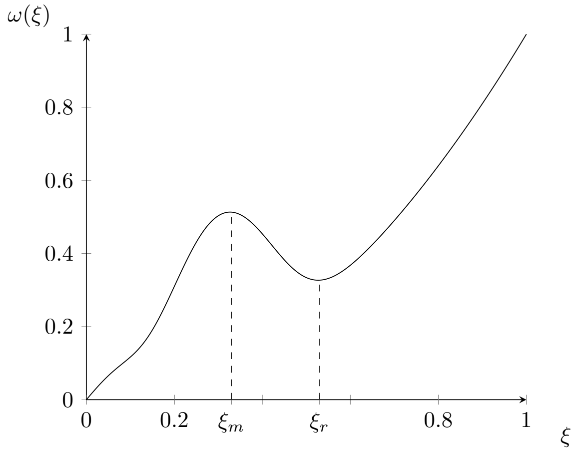


Figure 5.5: Dispersion curve associated with potential (5.24), with $a = -36$, $b = 2687$, $c = 30$. Here $\xi_m \sim 0.33$ and $\xi_r \sim 0.53$.

(H1) nor (H2) are satisfied. However, we observe in Figure 5.6 that the energy curve is still concave, and that the straight line $\sqrt{2}\mathfrak{q}$ is still a tangent to the curve. Moreover, we found the same critical value as before for the momentum, i.e. $\mathfrak{q}_* \sim 1.55$.

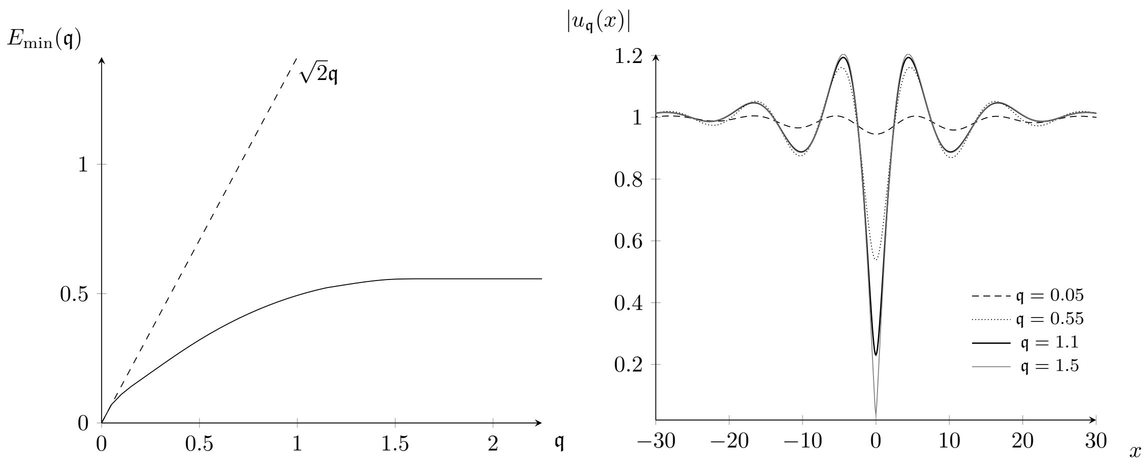


Figure 5.6: Curves $E_{\min}$ and solitons for the potential in (5.24), with $a = -36$, $b = 2687$, $c = 30$.

Chapter 6

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